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The MHD Theorems by Kaplan and Crocco and Their Consequences for MHD Flow

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Abstract

The MHD theorems by Kaplan and Crocco are re-derived and their consequences for analytical and numerical MHD calculations are discussed. It is demonstrated that the Kaplan theorem restricts the magnetic fields which admit potential flow and that the Crocco theorem describes entropy production due to vorticity. The consequences of these two theorems also for numerical codes are discussed.

1. Introduction

In these days various numerical fluid codes are in use in plasma physics. These codes use various systems of MHD equations. There are static codes for plasma containment devices and other codes consider flow equilibria. Equations of motion have been used for inviscid plasmas and other codes introduced an artificial or a Navier-Stokes viscosity.

The two theorems by Kaplan and by Crocco may help to investigate, if a system of MHD-equations is consistent or not. The Kaplan theorem describes the conditions to be satisfied that a plasma flow is a potential flow. The Crocco theorem connects plasma flow with thermodynamics. The most important quantity in thermodynamics is the entropy S. If a reversible process supplies heat $q_r = cdT$ to a thermodynamic system, then entropy S increases according to $dS = cdT/T$, where c is the (constant) polytropic specific heat and T designes temperature. The heat may be transformed into another form of energy and finally one will have $\oint dS = 0$ for reversible processes. All polytropic processes are reversible and for an ideal gas they are described by

$$p(\rho) = const \cdot \rho^n, \tag{1}$$

where

$$n = \frac{c - c_p}{c - c_V}. \tag{2}$$

c_p is the specific heat for isobaric and c_V for isochoric processes. For adiabatic (isentropic) processes ($c = 0$) (1) becomes the Poisson (adiabatic) law with $n = \gamma = c_p/c_V$.

For an irreversible process like dissipation by viscosity, the system may not be restored to its initial state without producing any change in the rest of the universe. Thus $\oint dS > 0$ will be valid for the system. A small viscosity may allow the generation of significant vorticity and completely destroy a potential flow. But even if the dissipation would only have a negligible effect on the entropy increase, the Crocco theorem demonstrates the entropy increase due to the vorticity of the flow.

Therefore, a caloric state equation (1) is valid for reversible processes only and can not be assumed for irreversible processes like viscous stresses or rotational flow. If the plasma exhibits viscosity η, electric conductivity σ and heat conductivity χ, then the energy theorem written per unit mass reads

$$\begin{aligned} \rho T\left(\frac{\partial S}{\partial t} + (\vec{v}\nabla)S\right) &\equiv \rho\frac{\partial U}{\partial t} + \rho(\vec{v}\nabla)U - \frac{p}{\rho}\frac{\partial \rho}{\partial t} - \frac{p}{\rho}(\vec{v}\nabla)\rho \\ &= \eta\sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\operatorname{div}\vec{v}\right)\frac{\partial v_i}{\partial x_k} + \frac{j^2}{\sigma} + \operatorname{div}(\chi\nabla T) \end{aligned} \tag{3}$$

and (1) has to be replaced by

$$p(\rho, S) = const \cdot \rho^\gamma R^\gamma \exp\left(\frac{S - S_0}{c_V}\right), \tag{4}$$

[9].

2. The Kaplan Theorem and Its Consequences

We now consider the flow of an ideal plasma ($\eta = 0$, $\chi = 0$, $\sigma = \infty$). In this case (3) becomes

$$\frac{dS}{dt} \equiv \frac{\partial S}{\partial t} + (\vec{v}\nabla)S = 0. \tag{5}$$

Some remarks concerning isentropy and Eq. (3) seem to be appropriate. Isentropy may be described by (5). For a steady flow this becomes $(\vec{v}\nabla)S = 0$. These two equations express that entropy is constant along a stream line. It should however be mentioned that the constant value of the entropy may differ for different fluid elements on different streamlines [9].

From (5) we thus conclude that the flow of an ideal plasma is isentropic and the equation of motion reads

$$\begin{aligned}\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} &\equiv \frac{\partial \vec{v}}{\partial t} + \nabla\frac{v^2}{2} - [\vec{v} \times \operatorname{curl}\vec{v}] \\ &= \frac{1}{\rho\mu_0}[\operatorname{curl}\vec{B} \times \vec{B}] - \frac{\nabla p}{\rho}.\end{aligned} \tag{6}$$

We now apply the operator curl on (6). Using the identity

$$\operatorname{curl}\left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho}\operatorname{curl}\nabla p + \nabla\left(\frac{1}{\rho}\right) \times \nabla p \tag{7}$$

we obtain from (6)

$$\frac{\partial}{\partial t}\operatorname{curl}\vec{v} - \operatorname{curl}[\vec{v} \times \operatorname{curl}\vec{v}] = \frac{1}{\mu_0}\operatorname{curl}\left[\frac{1}{\rho}\cdot\operatorname{curl}\vec{B} \times \vec{B}\right]. \tag{8}$$

The last rhs term in (7) vanishes, since due to the adiabatic law one has

$$\nabla\left(\frac{1}{\rho}\right) \times \nabla p = 0. \tag{9}$$

Assuming now an isentropic potential flow ($\operatorname{curl}\vec{v} = 0$), (8) gives the Kaplan theorem [2, 5, 10]

$$\operatorname{curl}\left(\frac{1}{\rho}[\operatorname{curl}\vec{B} \times \vec{B}]\right) = 0. \tag{10}$$

We thus conclude that isentropy is only a necessary but not sufficient criterion for the occurrence of potential flow. It might be of interest to mention that a similar condition like (10) but for $\rho = const$ is a consequence of the Kelvin theorem on the conservation of the circulation along closed field lines [8]. The condition (10) also implies a twodimensional flow $\vec{v}$ perpendicular to the magnetic field $\vec{B}$ [11]. The Kaplan condition is also satisfied by a forcefree magnetic field (Beltrami field) [6].

We furthermore conclude that the use of a caloric state equation of the type of Poisson's law has the consequence that the plasma flow is isentropic and can be potential only if (10) is satisfied. Such a potential flow is, however, restricted to special, i.e. apparently cases with straight field lines or to force-free fields. A further consequence would be that numerical codes including Poisson's law can not treat (non-forcefree) toroidal devices exactly.

3. The MHD Crocco Theorem and Its Consequences

If Navier-Stokes viscosity is taken into account when establishing a system of differential equations, the situation changes completely. The same is true for the assumption of an artificial friction term $-\alpha\vec{v}$, where α is a constant. Such models have been used by many authors ([1, 13] and others). It seems to be clear that viscous effects are irreversible and increase the entropy. Due to the entropy increase an equation $p(\rho)$ should not be used. In aerodynamics of ideal gases the well-known Crocco theorem connects vorticity curl $\vec{v}$ with entropy increase [7, 14]. For a compressible frictionless steady ($\partial/\partial t = 0$) gas flow defined by the equation of motion

$$(\vec{v}\nabla)\vec{v} = -\frac{1}{\rho}\nabla p, \tag{11}$$

the Crocco theorem reads

$$\vec{v} \times \mathrm{curl}\,\vec{v} = \nabla h - T\,\nabla S. \tag{12}$$

Here h is given by

$$h = U + \frac{p}{\rho} + \frac{1}{2}v^2. \tag{13}$$

Using a combination of the equation of motion (3) for a dissipative gas ($\eta \neq 0$, $\sigma = \infty$, $\chi = 0$) and the energy equation, Vazsonyi finds

$$\frac{dh}{dt} = \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\phi + \frac{\eta}{\rho}\vec{v}\left(\Delta\vec{v} + \frac{1}{3}\mathrm{grad}\ \mathrm{div}\,\vec{v}\right), \tag{14}$$

where ϕ is the dissipation function. For a steady frictionless gas flow it thus follows $h = const$ ('isoenergetic flow'). Due to the missing friction and due to the energy theorem one has $\nabla h = 0$ for an ideal isoenergetic gas. Thus the Crocco theorem for gasdynamics states that $\vec{v} \times \operatorname{curl} \vec{v} = -T\,\nabla S$. The importance of this equation lies in the fact that it relates the vorticity of the gas to the rates of change of the entropy S. Every irrotational steady isoenergetic flow of a gas must be isentropic and every anisentropic isoenergetic steady flow has vortices [12].

A similar situation should occur in plasma physics. The MHD Crocco theorem for a viscous plasma ($\eta \neq 0$, $\chi = 0$, $\sigma = \infty$) may be derived from the equation of motion (6) in the form including now viscosity

$$\nabla \frac{v^2}{2} - [\vec{v} \times \operatorname{curl} \vec{v}] = -\frac{\nabla p}{\rho} + \frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}] + \frac{\vec{R}}{\rho}. \tag{15}$$

The vector $\vec{R}$ designates any friction term. Now we use the second law of thermodynamics for a plasma with constant specific heat c_V

$$\nabla U + p\nabla \frac{1}{\rho} \equiv c_V \nabla T + p\nabla \frac{1}{\rho} = T\nabla S. \tag{16}$$

Adding (15) + (16) gives the MHD Crocco theorem [3, 4, 8]

$$\nabla h - \frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}] - [\vec{v} \times \operatorname{curl} \vec{v}] - \frac{\vec{R}}{\rho} = T\nabla S, \tag{17}$$

where h is again given by (13).

We now consider the consequences of the MHD Crocco theorem:

1. Isentropic flow: $\nabla S = 0$, therefore $\vec{R} = 0$ and $\nabla h = 0$. In order that this flow becomes a potential flow $(\operatorname{curl} \vec{v} = 0)$ or a Beltrami flow $\vec{v} || \operatorname{curl} \vec{v}$, the Kaplan theorem (10) must be satisfied.
2. Potential flow: $\operatorname{curl} \vec{v} = 0$, $\vec{R} = 0$, $\nabla h = 0$. From (17) we obtain

$$-\frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}] = T\nabla S. \tag{18}$$

We conclude that forcefree magnetic fields admit an isentropic potential flow. The application of the operator curl on the Crocco theorem delivers

$$\operatorname{curl}\left[\frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}]\right] = -\nabla T \times \nabla S. \tag{19}$$

We see that according to (10) a potential flow is possible only if

$$\text{a)}\nabla S = 0 \quad \text{or} \quad \text{b)}\nabla T = 0 \quad \text{or} \quad \text{c)}\nabla T \| \nabla S. \tag{20}$$

3. For an isoenergetic Beltrami flow $\vec{v} \times \operatorname{curl} \vec{v} = 0$ one has

$$-\frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}] = T\,\nabla S. \tag{21}$$

   In this case non-forcefree magnetic fields produce entropy.
4. For forcefree magnetic field curl $\vec{B} \times \vec{B} = 0$ with $\eta = 0$, $\vec{R} = 0$, one obtains (12).
5. Dissipative static equilibrium ($\vec{v} = 0$): from (13), (14) one has

$$\nabla\left(U + \frac{p}{\rho}\right) - \frac{1}{\rho\mu_0}[\operatorname{curl} \vec{B} \times \vec{B}] - \frac{\vec{R}}{\rho} = T\,\nabla S. \tag{22}$$

   Due to the dissipation, $\nabla S \neq 0$ and an equation $p = p(\rho)$ is not admitted and the equation of state $p = \rho RT$ as well as heat conduction have to be taken into account.

4. Conclusions

Exact analytical and accurate numerical calculations need a system of consistent equations. If not consistent equations like (1) are used or necessary conditions are not taken into account, the calculations will deliver wrong or inaccurate results only.

Based on these considerations, the main conclusions are:

1. Plasma processes which include dissipative effects can not be described by a state equation $p = p(\rho)$, but only by $p = p(\rho, T)$ or $p(\rho, S)$ etc.
2. If dissipative processes occur, temperature T and heat conduction must be included even for a plasma with infinite electric conductivity.
3. Numerical plasma codes which take into account dissipative effects are *inaccurate* and inconsistent, if an equation $p(\rho)$ is included in the system of equations on which the code is based.
4. Potential flow is isentropic, allows $p(\rho)$, but must satisfy the Kaplan condition. If this is not the case, no potential flow and no $p(\rho)$ can exist.
5. A numerical code which uses $p(\rho)$ can describe a potential flow and must have $(\vec{B}\nabla)\vec{B} = 0$. Thus 3D codes with curved non-forcefree magnetic field lines will be inaccurate.

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