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MULTIPLE HERMITE POLYNOMIALS AND SIMULTANEOUS GAUSSIAN QUADRATURE*

WALTER VAN ASSCHE† AND ANTON VUERINCKX†

Dedicated to Walter Gautschi on the occasion of his 90th birthday

Abstract. Multiple Hermite polynomials are an extension of the classical Hermite polynomials for which orthogonality conditions are imposed with respect to $r > 1$ normal (Gaussian) weights $w_j(x) = e^{-x^2+c_jx}$ with different means $c_j/2$, $1 \leq j \leq r$. These polynomials have a number of properties such as a Rodrigues formula, recurrence relations (connecting polynomials with nearest neighbor multi-indices), a differential equation, etc. The asymptotic distribution of the (scaled) zeros is investigated, and an interesting new feature is observed: depending on the distance between the c_j, $1 \leq j \leq r$, the zeros may accumulate on s disjoint intervals, where $1 \leq s \leq r$. We will use the zeros of these multiple Hermite polynomials to approximate integrals of the form $\int_{-\infty}^{\infty} f(x)\exp(-x^2+c_jx)\,dx$ simultaneously for $1 \leq j \leq r$ for the case $r = 3$ and the situation when the zeros accumulate on three disjoint intervals. We also give some properties of the corresponding quadrature weights.

Key words. multiple Hermite polynomials, simultaneous Gauss quadrature, zero distribution, quadrature coefficients

AMS subject classifications. 33C45, 41A55, 42C05, 65D32

1. Simultaneous Gauss quadrature.

Let $w_1, \ldots, w_r$ be $r \geq 1$ weight functions on $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Simultaneous quadrature is a numerical method to approximate r integrals

$$\int_{\mathbb{R}} f(x)w_j(x)\,dx, \qquad 1 \leq j \leq r,$$

by sums

$$\sum_{k=1}^{N} \lambda_{k,N}^{(j)} f(x_{k,N})$$

at the same N points $\{x_{k,N}, 1 \leq k \leq N\}$ but with weights $\{\lambda_{k,N}^{(j)}, 1 \leq k \leq N\}$ which depend on j. This was described by Borges [3] in 1994 but was originally suggested by Aurel Angelescu [1] in 1918, whose work seems to have gone unnoticed. The past few decades, it became clear that this is closely related to multiple orthogonal polynomials in a similar way as Gaussian quadrature is related to orthogonal polynomials. The motivation in [3] involved a color signal f, which can be transmitted using three colors: red-green-blue (RGB). For this we need the amount of R-G-B in f given by

$$\int_{-\infty}^{\infty} f(x)w_R(x)\,dx, \qquad \int_{-\infty}^{\infty} f(x)w_G(x)\,dx, \qquad \int_{-\infty}^{\infty} f(x)w_B(x)\,dx.$$

A natural question is whether this can be calculated with N function evaluations and a maximum degree of accuracy. If we choose n points for each integral and then use Gaussian quadrature, then this would require $3n$ function evaluations for a degree of accuracy of $2n-1$. A better choice is to choose the zeros of the multiple orthogonal polynomials $P_{n,n,n}$ for the

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†Department of Mathematics, KU Leuven, Celestijnenlaan 200B box 2400, BE-3001 Leuven, Belgium (walter.vanassche@kuleuven.be).

weights (w_R, w_G, w_B) and then use interpolatory quadrature. This again requires $3n$ function evaluations, but the degree of accuracy is increased to $4n-1$. We will call this method based on the zeros of multiple orthogonal polynomials the simultaneous Gaussian quadrature method. Some interesting research problems for simultaneous Gaussian quadrature are:

- To find the multiple orthogonal polynomials when the weights $w_1, \ldots, w_r$ are given.
- To study the location and computation of the zeros of the multiple orthogonal polynomials.
- To study the behavior and computation of the weights $\lambda_{k,N}^{(j)}$.
- To investigate the convergence of the quadrature rules.

Part of this research has already been started in [4, 5, 8, 9, 13], but there is still a lot to be done in this field.

2. Multiple orthogonal polynomials.

Let us introduce multiple orthogonal polynomials.

DEFINITION 2.1. *Let $\mu_1, \ldots, \mu_r$ be r positive measures on $\mathbb{R}$, and let $\vec{n} = (n_1, \ldots, n_r)$ be a multi-index in $\mathbb{N}^r$. The (type II) multiple orthogonal polynomial $P_{\vec{n}}$ is the monic polynomial of degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ that satisfies the orthogonality conditions*

$$\int x^k P_{\vec{n}}(x)\, d\mu_j(x) = 0, \qquad 0 \le k \le n_j - 1,$$

for $1 \le j \le r$.

Such a monic polynomial may not exist or may not be unique. One needs conditions on (the moments of) the measures $(\mu_1, \ldots, \mu_r)$. Two important cases have been introduced for which all the multiple orthogonal polynomials exist and are unique. The measures $(\mu_1, \ldots, \mu_r)$ are an *Angelesco system* if $\text{supp}(\mu_j) \subset \Delta_j$, where the Δ_j are intervals which are pairwise disjoint: $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$.

THEOREM 2.2 (Angelesco). *For an Angelesco system the multiple orthogonal polynomials exist for every multi-index $\vec{n}$. Furthermore, $P_{\vec{n}}$ has n_j simple zeros in each interval Δ_j.*

For a proof, see [10, Chapter 4, Proposition 3.3] or [6, Theorem 23.1.3]. The behavior of the quadrature weights for simultaneous Gaussian quadrature is known for this case (see [10, Chapter 4, Proposition 3.5], [8, Theorem 1.1]):

THEOREM 2.3. *The quadrature weights $\lambda_{k,n}^{(j)}$ are positive for the n_j zeros on Δ_j. The remaining quadrature weights have alternating sign with those for the zeros closest to the interval Δ_j being positive.*

Another important case is when the measures form an AT-system. The weight functions $(w_1, \ldots, w_r)$ are an *algebraic Chebyshev system* (AT-system) on $[a, b]$ if

$$\begin{aligned} &w_1, xw_1, x^2w_1, \ldots, x^{n_1-1}w_1,\ w_2, xw_2, x^2w^2, \ldots, x^{n_2-1}w_2, \ldots, \\ &w_r, xw_r, x^2w_r, \ldots, x^{n_r-1}w_r \end{aligned}$$

are a Chebyshev system on $[a, b]$ for every $(n_1, \ldots, n_r) \in \mathbb{N}^r$.

THEOREM 2.4. *For an AT-system the multiple orthogonal polynomials exist for every multi-index $(n_1, \ldots, n_r)$. Furthermore, $P_{\vec{n}}$ has $|\vec{n}|$ simple zeros on the interval $[a, b]$.*

For a proof, see [10, Chapter 4, Corollary of Theorem 4.3] or [6, Theorem 23.1.4].

3. Multiple Hermite polynomials. We will consider the weight functions

$$w_j(x) = e^{-x^2+c_jx}, \qquad x \in \mathbb{R},$$

with real parameters $c_1, \ldots, c_j$ such that $c_i \neq c_j$ whenever $i \neq j$. These weights are proportional to normal weights with means at $c_j/2$ and variance $\sigma^2 = \frac{1}{2}$. They form an AT-system, and the corresponding multiple orthogonal polynomials are known as *multiple Hermite polynomials* $H_{\vec{n}}$. They can be obtained by using the Rodrigues formula

$$e^{-x^2} H_{\vec{n}}(x) = (-1)^{|\vec{n}|} 2^{-|\vec{n}|} \left(\prod_{j=1}^{r} e^{-c_jx} \frac{d^{n_j}}{dx^{n_j}} e^{c_jx} \right) e^{-x^2},$$

from which one can find the explicit expression

$$H_{\vec{n}}(x) = (-1)^{|\vec{n}|} 2^{-|\vec{n}|} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} c_1^{n_1-k_1} \cdots c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x);$$

see [6, Section 23.5]. Multiple orthogonal polynomials satisfy a system of recurrence relations connecting the nearest neighbors. For multiple Hermite polynomials one has

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x), \qquad 1 \le k \le r,$$

where $\vec{e}_1 = (1, 0, 0, \ldots, 0), \vec{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \vec{e}_r = (0, 0, \ldots, 0, 1)$. They also have interesting differential relations such as r raising operators

$$\left(e^{-x^2+c_jx} H_{\vec{n}-\vec{e}_j}(x) \right)' = -2e^{-x^2+c_jx} H_{\vec{n}}(x), \qquad 1 \le j \le r,$$

and a lowering operator

$$H'_{\vec{n}}(x) = \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x);$$

see [6, Equations (23.8.5)–(23.8.6)]. Combining these raising operators and the lowering operator gives a differential equation of order $r + 1$,

$$\left(\prod_{j=1}^{r} D_j \right) DH_{\vec{n}}(x) = -2 \left(\sum_{j=1}^{r} n_j \prod_{i \neq j} D_j \right) H_{\vec{n}}(x),$$

where

$$D = \frac{d}{dx}, \qquad D_j = e^{x^2-c_jx} D e^{-x^2+c_jx} = D + (-2x + c_j)I.$$

From now on we deal with the case $r = 3$ and weights $c_1 = -c, c_2 = 0, c_3 = c$:

$$w_1(x) = e^{-x^2-cx}, \qquad w_2(x) = e^{-x^2}, \qquad w_3(x) = e^{-x^2+cx}.$$

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3.1. Zeros. Let $x_{1,3n} < \ldots < x_{3n,3n}$ be the zeros of $H_{n,n,n}$. First we will show that for c large enough, the zeros of $H_{n,n,n}$ lie on three disjoint intervals around $-c/2$, 0, and $c/2$.

PROPOSITION 3.1. *For c sufficiently large (e.g., $c > 4\sqrt{4n+1}$) the zeros of $H_{n,n,n}$ are in three disjoint intervals $I_1 \cup I_2 \cup I_3$, where*

$$I_1 = \left[-\frac{c}{2} - \sqrt{4n+1}, -\frac{c}{2} + \sqrt{4n+1}\right], \qquad I_2 = \left[-\sqrt{4n+1}, \sqrt{4n+1}\right],$$
$$I_3 = \left[\frac{c}{2} - \sqrt{4n+1}, \frac{c}{2} + \sqrt{4n+1}\right],$$

and each interval contains n simple zeros.

Proof. Suppose $x_1, x_2, \ldots, x_m$ are the sign changes of $H_{n,n,n}$ on I_3 and that $m < n$. Let $\pi_m(x) = (x - x_1)(x - x_2) \cdots (x - x_m)$. Then $H_{n,n,n}(x)\pi_m(x)$ does not change sign on I_3. By the multiple orthogonality one has

$$\int_{-\infty}^{\infty} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx = 0. \tag{3.1}$$

Suppose that $H_{n,n,n}(x)\pi_m(x)$ is positive on I_3. Then by the infinite-finite range inequalities (see, e.g., [7, Chapter 4, Theorem 4.1], where we take $Q(x) = x^2 - cx$, $p = 1$, $t = 4n + 1$ so that $\Delta_t = I_3$), one has

$$\int_{\mathbb{R}\setminus I_3} |H_{n,n,n}(x)\pi_m(x)|e^{-x^2+cx}\,dx < \int_{I_3} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx$$

so that

$$\int_{\mathbb{R}\setminus I_3} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx > -\int_{I_3} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx$$

and

$$\int_{-\infty}^{\infty} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx$$
$$= \int_{I_3} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx + \int_{\mathbb{R}\setminus I_3} H_{n,n,n}(x)\pi_m(x)e^{-x^2+cx}\,dx > 0,$$

which is in contradiction with (3.1). This means that our assumption that $m < n$ is false, and hence $m \geq n$. We can repeat the reasoning for I_2 and I_1, and since $H_{n,n,n,}$ is a polynomial of degree $3n$, we must conclude that each interval contains n zeros of $H_{n,n,n}$, which are all simple. Clearly the three intervals are disjoint when $c > 4\sqrt{4n+1}$. ∎

This result shows that for large c, the multiple Hermite polynomials behave very much like an Angelesco system, i.e., multiple orthogonal polynomials for which the orthogonality conditions are on disjoint intervals. Some results for simultaneous Gauss quadrature for Angelesco systems were proved earlier in [10, Chapter 4, Propositions 3.4 and 3.5] and [8]. In this paper we will show that similar results are true for multiple Hermite polynomials when c is large.

The intervals I_1, I_2, I_3 are in fact a bit too large because they were obtained by using the infinite-finite range inequalities for one weight only and not for the three weights simultaneously. In order to study the zeros in more detail, we will take the parameter c proportional to $\sqrt{n}$ and scale the zeros by a factor $\sqrt{n}$ as well. This amounts to investigating the polynomials $H_{n,n,n}(\sqrt{n}x)$ with $c = \sqrt{n}\hat{c}$. In order to find for which values of c the zeros are accumulating

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on three disjoint intervals as $n \to \infty$, we investigate the asymptotic distribution of the zeros. Our main theorem in this section is

THEOREM 3.2. *There exists a $c^* > 0$ such that for the zeros of $H_{n,n,n}$ with $c = \sqrt{n}\hat{c}$ one has*

$$\lim_{n\to\infty} \frac{1}{3n} \sum_{j=1}^{3n} f\left(\frac{x_{j,3n}}{\sqrt{n}}\right) = \int f(x)v(x)\,dx,$$

where v is a probability density supported on three intervals $[-b,-a] \cup [-d,d] \cup [a,b]$ $(0 < d < a < b)$ when $\hat{c} > c^*$, and v is supported on one interval $[-b,b]$ when $\hat{c} < c^*$. The numerical value is $c^* = 4.10938818$.

Such a phase transition when the zeros cluster on one interval when the parameters are close together or on two intervals when the parameters are far apart was first observed and proved for $r = 2$ by Bleher and Kuijlaars [2].

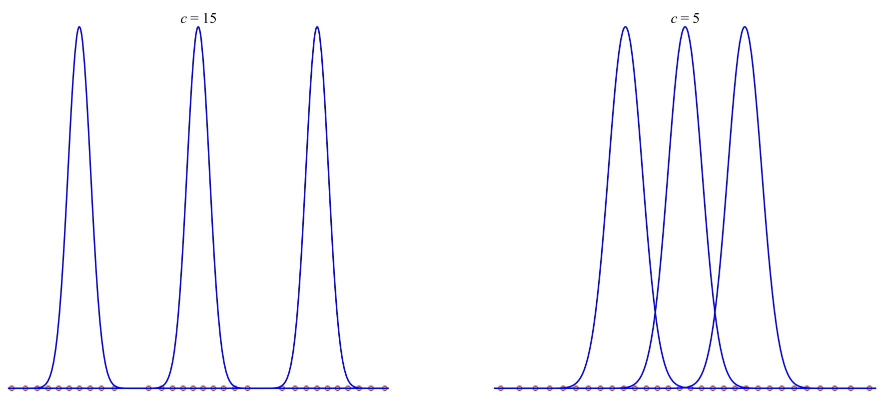

FIG. 3.1. *The weight functions and the zeros of $H_{10,10,10}$ for $c = 15$ (left) and $c = 5$ (right).*

Proof. The differential equation for $y = H_{n,n,n}(x)$ becomes

$$\begin{aligned} y'''' - 6xy''' + (12x^2 - c^2 - 6)y'' &+ [-8x^3 + (2c^2 + 12)x]y' \\ &= -2n[3y'' - 12xy' + (12x^2 - c^2 - 6)y]. \end{aligned}$$

The scaling amounts to studying zeros of $H_{n,n,n}(\sqrt{n}x)$, and these are multiple orthogonal polynomials for the weight functions

$$w_1(x) = e^{-n(x^2+\hat{c}x)}, \qquad w_2(x) = e^{-nx^2}, \qquad w_3(x) = e^{-n(x^2-\hat{c}x)}.$$

Consider the rational function

$$S_n(z) = \frac{1}{\sqrt{n}} \frac{H'_{n,n,n}(\sqrt{n}z)}{H_{n,n,n}(\sqrt{n}z)} = \frac{1}{n} \sum_{j=1}^{3n} \frac{1}{z - \frac{x_{j,3n}}{\sqrt{n}}} = \int \frac{d\mu_n(x)}{z - x},$$

where μ_n is the discrete measure with mass $1/n$ at each scaled zero $x_{j,3n}/\sqrt{n}$:

$$\mu_n = \frac{1}{n} \sum_{j=1}^{3n} \delta_{x_{j,3n}/\sqrt{n}}.$$

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The sequence $(S_n)_{n\in\mathbb{N}}$ is a family of analytic functions which is uniformly bounded on every compact subset of $\mathbb{C}\setminus\mathbb{R}$, hence by Montel's theorem there exists a subsequence $(S_{n_k})_k$ that converges uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$ to an analytic function S, and also its derivatives converge uniformly on these compact subsets:

$$S_{n_k}\to S,\qquad S'_{n_k}\to S',\qquad S''_{n_k}\to S'',\qquad S'''_{n_k}\to S'''.$$

Since each S_n is a Stieltjes transform of a positive measure (with total mass 3), the limit is of the form

$$S(z)=3\int\frac{d\mu(x)}{z-x}\,dx$$

with μ a probability measure on $\mathbb{R}$ that describes the asymptotic distribution of the scaled zeros, and μ_n converges weakly to the measure 3μ for the chosen subsequence. This function S may depend on the selected subsequence $(n_k)_k$, but we will show that every convergent subsequence has the same limit S. Observe that

$$H'_{n,n,n}(\sqrt{n}z)=\sqrt{n}S_nH_{n,n,n}(\sqrt{n}z),$$

from which we can find

$$\begin{aligned}
H''_{n,n,n}(\sqrt{n}z)&=(S'_n+nS_n^2)H_{n,n,n}(\sqrt{n}z),\\
H'''_{n,n,n}(\sqrt{n}z)&=\frac{1}{\sqrt{n}}(S''_n+3nS'_nS_n+n^2S_n^3)H_{n,n,n}(\sqrt{n}z),\\
H''''_{n,n,n}(\sqrt{n}z)&=\frac{1}{n}(S'''_n+4nS''_nS_n+3n(S'_n)^2+6n^2S_n^2S'_n+n^3S_n^4)H_{n,n,n}(\sqrt{n}z).
\end{aligned}$$

Put this into the differential equation (with $x=\sqrt{n}z$ and $c=\sqrt{n}\hat{c}$). Then as $n=n_k\to\infty$ one finds

$$S^4-6zS^3+(12z^2-\hat{c}^2+6)S^2+(-8z^3+2\hat{c}^2z-24z)S+2(12z^2-\hat{c}^2)=0. \tag{3.2}$$

This is an algebraic equation of order 4, and hence it has four solutions $S_{(1)},S_{(2)},S_{(3)},S_{(4)}$. A careful analysis of these solutions and equation (3.2) near infinity shows that for $z\to\infty$,

$$\begin{aligned}
S_{(1)}(z)&=\frac{3}{z}+\mathcal{O}(\frac{1}{z^2}), &\qquad S_{(2)}(z)&=2z+\hat{c}+\mathcal{O}(\frac{1}{z}),\\
S_{(3)}(z)&=2z+\mathcal{O}(\frac{1}{z}), &\qquad S_{(4)}(z)&=2z-\hat{c}+\mathcal{O}(\frac{1}{z}).
\end{aligned}$$

We are therefore interested in $S_{(1)}(z)$ since it gives the required Stieltjes transform

$$S_{(1)}(z)=S(z)=3\int\frac{d\mu(x)}{z-x}\,dx.$$

The algebraic equation is independent of the selected subsequence, which implies that every subsequence $(S_{n_k})_k$ has the same limit, which in turn implies that the full sequence $(S_n)_n$ converges to this limit S. The measure μ can be retrieved by using the Stieltjes-Perron inversion theorem

$$\mu((a,b))+\frac{1}{2}\mu(\{a\})+\mu(\{b\})=\lim_{\epsilon\to0+}\frac{1}{2\pi i}\int_a^b\frac{S(x-i\epsilon)-S(x+i\epsilon)}{3}\,dx.$$

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If μ has no mass points, then the density v of μ is given by

$$v(x) = \frac{1}{2\pi i} \lim_{\epsilon \to 0+} \frac{S(x - i\epsilon) - S(x + i\epsilon)}{3}.$$

Hence the support of the density v is given by the set on $\mathbb{R}$ where S has a jump discontinuity. This can be analyzed by investigating the discriminant of the algebraic expression

$$\begin{aligned} &256\hat{c}^6 z^6 - 128\hat{c}^4(\hat{c}^4 + 18\hat{c}^2 - 18)z^4 \\ &\quad + 16\hat{c}^2(\hat{c}^8 + 12\hat{c}^6 + 240\hat{c}^4 - 1008\hat{c}^2 + 432)z^2 \\ &\quad - 32\hat{c}^2(\hat{c}^2 + 4\hat{c} + 6)^2(\hat{c}^2 - 4\hat{c} + 6)^2. \end{aligned} \tag{3.3}$$

This is a polynomial of degree 6 in the variable z. The support of v is where this polynomial is negative. There is a phase transition from one interval to three intervals when the z-polynomial (3.3) has two double roots. This happens when the discriminant of the z-polynomial (3.3) is zero:

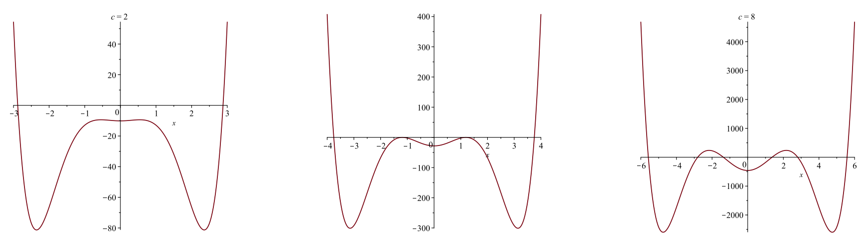

FIG. 3.2. *The polynomial* (3.3) *for* $\hat{c} = 2$ *(left),* $\hat{c} = \hat{c}^*$ *(middle), and* $\hat{c} = 8$ *(right).*

$$(\hat{c}^2 - 4\hat{c} + 6)^2 \hat{c}^{32} (\hat{c}^2 + 2)^4 (\hat{c}^2 + 4\hat{c} + 6)^2 (\hat{c}^6 - \frac{27}{2}\hat{c}^4 - 54\hat{c}^2 - 54)^6 = 0.$$

The only positive real zero is the positive real root of

$$\hat{c}^6 - \frac{27}{2}\hat{c}^4 - 54\hat{c}^2 - 54 = 0,$$

and this is $c^* = 4.10938818$. □

Observe that the phase transition c^* is at a smaller value than the one suggested by Proposition 3.1, which would give the value 8. As mentioned before, this is because in Proposition 3.1 we used the infinite-finite range inequalities for one single weight and not for the three weights simultaneously.

4. Some potential theory. From now one we assume that $\hat{c} > c^* = 4.10938818$. The Stieltjes transform of the asymptotic zero distribution is

$$3 \int \frac{v(x)}{z - x}\, dx = S(z) = \int_{-b}^{-a} \frac{d\nu_1(x)}{z - x} + \int_{-d}^{d} \frac{d\nu_2(x)}{z - x} + \int_{a}^{b} \frac{d\nu_3(x)}{z - x}.$$

The measures ν_1, ν_2, ν_3 are unit measures that are minimizing the expression

$$\sum_{i=1}^{3} \sum_{j=1}^{3} c_{i,j} I(\mu_i, \mu_j) + \sum_{i=1}^{3} \int V_i(x)\, d\mu_i(x)$$

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over all unit measures μ_1, μ_2, μ_3 supported on $\mathbb{R}$, with

$$I(\mu_i, \mu_j) = \iint \log \frac{1}{|x-y|}\, d\mu_i(x)\, d\mu_j(x), \qquad C = (c_{i,j}) = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

and

$$V_1(x) = x^2 + \hat{c}x, \qquad V_2(x) = x^2, \qquad V_3(x) = x^2 - \hat{c}x.$$

This is the vector equilibrium problem for an Angelesco system [10, Chapter 5, Section 6]. Define the logarithmic potential

$$U(x;\mu) = \int \log \frac{1}{|x-y|}\, d\mu(y).$$

The variational conditions for this vector equilibrium problem are

$$2U(x;\nu_1) + U(x;\nu_2) + U(x;\nu_3) + V_1(x) = \ell_1, \qquad x \in [-b,-a], \tag{4.1}$$
$$2U(x;\nu_1) + U(x;\nu_2) + U(x;\nu_3) + V_1(x) \geq \ell_1, \qquad x \in \mathbb{R} \setminus [-b,-a], \tag{4.2}$$

$$U(x;\nu_1) + 2U(x;\nu_2) + U(x;\nu_3) + V_2(x) = \ell_2, \qquad x \in [-d,d], \tag{4.3}$$
$$U(x;\nu_1) + 2U(x;\nu_2) + U(x;\nu_3) + V_2(x) \geq \ell_2, \qquad x \in \mathbb{R} \setminus [-d,d], \tag{4.4}$$

$$U(x;\nu_1) + U(x;\nu_2) + 2U(x;\nu_3) + V_3(x) = \ell_3, \qquad x \in [a,b], \tag{4.5}$$
$$U(x;\nu_1) + U(x;\nu_2) + 2U(x;\nu_3) + V_3(x) \geq \ell_3, \qquad x \in \mathbb{R} \setminus [a,b], \tag{4.6}$$

where ℓ_1, ℓ_2, ℓ_3 are constants (Lagrange multipliers). As an example, we have plotted these functions in Figure 4.1 for $\hat{c} = 6$. The measures ν_1, ν_2, ν_3 give the asymptotic distribution of

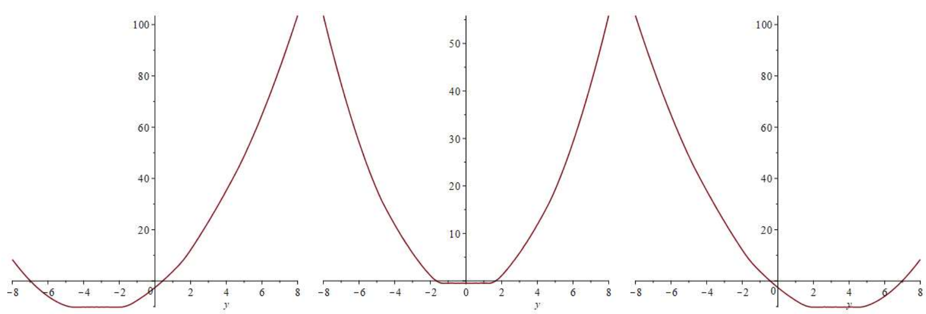

FIG. 4.1. $2U(x;\nu_1) + U(x;\nu_2) + U(x;\nu_3) + V_1(x)$ *(left)*, $U(x;\nu_1) + 2U(x;\nu_2) + U(x;\nu_3) + V_2(x)$ *(middle)*, $U(x;\nu_1) + U(x;\nu_2) + 2U(x;\nu_3) + V_3(x)$ *(right)*.

the (scaled) zeros of $H_{n,n,n}$ on the intervals $[-b,-a]$, $[-d,d]$, and $[a,b]$, respectively. They are absolutely continuous, and their densities can be found from the jumps of an algebraic function ξ on the real line. The function ξ satisfies the algebraic equation

$$\xi^4 - 2z\hat{c}\xi^3 + (6-\hat{c}^2)\xi^2 + 2\hat{c}^2 z\xi - 2\hat{c}^2 = 0,$$

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which has four solutions $\xi_1, \xi_2, \xi_3, \xi_4$, which behave near infinity as

$$\xi_1(z) = 2z - \frac{3}{z} + \mathcal{O}(\frac{1}{z^2}), \qquad \xi_2(z) = -\hat{c} + \frac{1}{z} + \mathcal{O}(\frac{1}{z^2}),$$
$$\xi_3(z) = \frac{1}{z} + \mathcal{O}(\frac{1}{z^2}), \qquad \xi_4(z) = \hat{c} + \frac{1}{z} + \mathcal{O}(\frac{1}{z^2}).$$

The densities ν_1', ν_2', ν_3' are given by

$$\nu_1'(x) = -\frac{(\xi_2)_+(x) - (\xi_2)_-(x)}{2\pi i}, \qquad \nu_2'(x) = -\frac{(\xi_3)_+(x) - (\xi_3)_-(x)}{2\pi i},$$
$$\nu_3'(x) = -\frac{(\xi_4)_+(x) - (\xi_4)_-(x)}{2\pi i}.$$

The relation between the algebraic function S from (3.2) is given by

$$S = \frac{2}{\xi} + \frac{2}{\xi + \hat{c}} + \frac{2}{\xi - \hat{c}}.$$

The Stieltjes transforms of ν_1, ν_2, ν_3 are related to the solutions of (3.2) by

$$S_{(1)}(z) = \int_{-b}^{-a} \frac{d\nu_1(x)}{z-x} + \int_{-d}^{d} \frac{d\nu_2(x)}{z-x} + \int_a^b \frac{d\nu_3(x)}{z-x}, \quad S_{(3)}(z) = 2z - \int_{-d}^{d} \frac{d\nu_2(x)}{z-x},$$
$$S_{(2)}(z) = 2z + \hat{c} - \int_{-b}^{-a} \frac{d\nu_1(x)}{z-x}, \qquad S_{(4)}(z) = 2z - \hat{c} - \int_a^b \frac{d\nu_3(x)}{z-x}.$$

5. The quadrature weights. Recall that for polynomials f of degree $\leq 4n-1$

$$\int_{-\infty}^{\infty} f(x) e^{-n(x^2+\hat{c}x)}\, dx = \sum_{k=1}^{3n} \lambda_{k,3n}^{(1)} f(x_{k,3n}), \tag{5.1}$$

$$\int_{-\infty}^{\infty} f(x) e^{-nx^2}\, dx = \sum_{k=1}^{3n} \lambda_{k,3n}^{(2)} f(x_{k,3n}), \tag{5.2}$$

$$\int_{-\infty}^{\infty} f(x) e^{-n(x^2-\hat{c}x)}\, dx = \sum_{k=1}^{3n} \lambda_{k,3n}^{(3)} f(x_{k,3n}). \tag{5.3}$$

Here $x_{k,3n}$ are the zeros of $H_{n,n,n}(x) = p_n(x)q_n(x)r_n(x)$, where p_n has its zeros on $[-b,-a]$, q_n on $[-d,d]$, and r_n on $[a,b]$. Take $f(x) = \pi_{2n-1}(x)q_n(x)r_n(x)$ with π_{2n-1} of degree $\leq 2n-1$. Then (5.1) gives

$$\int_{-\infty}^{\infty} \pi_{2n-1}(x) q_n(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx = \sum_{k=1}^{n} \lambda_{k,3n}^{(1)} q_n(x_k) r_n(x_k) \pi_{2n-1}(x_k).$$

This is the Gaussian quadrature formula for the weight function $q_n(x)r_n(x)e^{-n(x^2+\hat{c}x)}$ with quadrature nodes at the zeros of p_n. So we have

LEMMA 5.1. *The first n quadrature weights for the first integral (5.1) are*

$$\lambda_{k,3n}^{(1)} q_n(x_{k,3n}) r_n(x_{k,3n}) = \lambda_{k,n}(q_n r_n\, d\mu_1), \qquad 1 \leq k \leq n,$$

where $\lambda_{k,n}(q_n r_n\, d\mu_1)$ are the usual Christoffel numbers of Gaussian quadrature for the weight $q_n(x)p_n(x)e^{-n(x^2+\hat{c}x)}$ on $\mathbb{R}$.

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Johann Radon Institute (RICAM)

For the middle n quadrature weights and the last n quadrature weights, we have a weaker statement. By taking $f(x) = \pi_{n-1}(x)p_n^2(x)r_n(x)$ with π_{n-1} of degree $\leq n-1$, the quadrature formula (5.1) gives

$$\int_{-\infty}^{\infty} \pi_{n-1}(x)p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx = \sum_{k=n+1}^{2n} \lambda_{k,3n}^{(1)} p_n^2(x_k)r_n(x_k)\pi_{n-1}(x_k).$$

This is not a Gaussian quadrature rule but the Lagrange interpolatory rule for the weight function $p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}$ with quadrature nodes at the zeros of q_n. So now we have the result:

LEMMA 5.2. *The middle n quadrature weights for the first integral are*

$$\lambda_{k,3n}^{(1)} p_n^2(x_{k,3n})r_n(x_{k,3n}) = w_{k,n}(q_n), \qquad n+1 \leq k \leq 2n,$$

where $w_{k,n}(q_n)$ are the quadrature weights for the Lagrange interpolatory quadrature at the zeros of q_n and the weight function $p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}$.

In a similar way, we take $f(x) = \pi_{n-1}(x)p_n^2(x)q_n(x)$ with π_{n-1} of degree $\leq n-1$ so that (5.1) becomes

$$\int_{-\infty}^{\infty} \pi_{n-1}(x)p_n^2(x)q_n(x)e^{-n(x^2+\hat{c}x)}\,dx = \sum_{k=2n+1}^{3n} \lambda_{k,3n}^{(1)} p_n^2(x_k)q_n(x_k)\pi_{n-1}(x_k).$$

We then have:

LEMMA 5.3. *The last n quadrature weights for the first integral are*

$$\lambda_{k,3n}^{(1)} p_n^2(x_{k,3n})q_n(x_{k,3n}) = w_{k,n}(r_n), \qquad 2n+1 \leq k \leq 3n,$$

where $w_{k,n}(r_n)$ are the quadrature weights for the Lagrange interpolatory quadrature at the zeros of r_n and the weight function $p_n^2(x)q_n(x)e^{-n(x^2+\hat{c}x)}$.

Of course similar results are true for the quadrature weights $\lambda_{k,3n}^{(2)}$ for the second integral (5.2) and the quadrature weights $\lambda_{k,3n}^{(3)}$ for the third integral (5.3).

The weight function $q_n(x)r_n(x)e^{-n(x^2+\hat{c}x)}$ is not a positive weight on the whole real line, but it is positive on $[-b,-a]$ since the zeros of q_n and r_n are on $[-d,d]$ and $[a,b]$, respectively, at least when n is large. We can prove the following result.

THEOREM 5.4. *Let $\hat{c}$ be sufficiently large.*[1] *For the quadrature weights of the first integral* (5.1)*, one has*

$$\lambda_{k,3n}^{(1)} > 0, \qquad 1 \leq k \leq n,$$

and

$$\operatorname{sign} \lambda_{k,3n}^{(1)} = (-1)^{k-n+1}, \qquad n+1 \leq k \leq 3n.$$

Proof. For the first n weights we use $f(x) = p_n^2(x)q_n(x)r_n(x)/(x-x_{k,3n})^2$ in (5.1) to find (we write $x_k = x_{k,3n}$)

$$\lambda_{k,3n}^{(1)} [p_n'(x_k)]^2 q_n(x_k)r_n(x_k) = \int_{-\infty}^{\infty} \frac{p_n^2(x)}{(x-x_k)^2} q_n(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx.$$

[1] $\hat{c} > 8$ certainly works, but we conjecture that $\hat{c} > c^*$ is sufficient.

Clearly $[p_n'(x_k)]^2 q_n(x_k) r_n(x_k) > 0$ since $x_k \in [-b,-a]$ and the zeros of q_n and r_n are on $[-d,d]$ and $[a,b]$, respectively. So we need to prove that the integral is positive. Let $I_1 = [-\frac{\hat{c}}{2} - \sqrt{4+1/n}, -\frac{\hat{c}}{2} + \sqrt{4+1/n}]$. Then by Proposition 3.1 all the zeros of p_n are in I_1, and hence $[-b,-a] \subset I_1$. If $\hat{c}$ is large enough, then $q_n r_n$ is positive on I_1, and by the infinite-finite range inequality (see Proposition 3.1)

$$\int_{\mathbb{R}\setminus I_1} \frac{p_n^2(x)}{(x-x_k)^2} |q_n(x) r_n(x)| e^{-n(x^2+\hat{c}x)}\, dx > \int_{I_1} \frac{p_n^2(x)}{(x-x_k)^2} q_n(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx,$$

so that

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{p_n^2(x)}{(x-x_k)^2} q_n(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx \\ &= \int_{I_1} \frac{p_n^2(x) q_n(x) r_n(x)}{(x-x_k)^2} e^{-n(x^2+\hat{c}x)}\, dx + \int_{\mathbb{R}\setminus I_1} \frac{p_n^2(x) q_n(x) r_n(x)}{(x-x_k)^2} e^{-n(x^2+\hat{c}x)}\, dx > 0. \end{aligned}$$

For the middle n quadrature weights we use Lemma 5.2. Clearly $p_n^2(x_k) > 0$ and sign $r_n(x_k) = (-1)^n$ since all the zeros of r_n are on the interval $[a,b]$ and $x_k \in [-d,d]$ for $n+1 \leq k \leq 2n$. Furthermore for the Lagrange quadrature nodes one has

$$w_{k,n}(q_n) = \int_{-\infty}^{\infty} \frac{q_n(x)}{(x-x_k) q_n'(x_k)} p_n^2(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx,$$

where sign $q_n'(x_k) = (-1)^{k-2n}$. Observe that for a large enough parameter $\hat{c}$ one obtains sign $q_n(x)/(x-x_k) = (-1)^{n-1}$ on I_1 since all the zeros of q_n are on $[-d,d]$ and also sign $r_n(x) = (-1)^n$ on I_1 since all the zeros of r_n are on $[a,b]$. By the infinite-finite range inequality one has

$$\int_{\mathbb{R}\setminus I_1} \frac{|q_n(x)|}{|x-x_k|} p_n^2(x) |r_n(x)| e^{-n(x^2+\hat{c}x)}\, dx < -\int_{I_1} \frac{q_n(x)}{x-x_k} p_n^2(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx$$

so that

$$\int_{-\infty}^{\infty} \frac{q_n(x)}{(x-x_k)} p_n^2(x) r_n(x) e^{-n(x^2+\hat{c}x)}\, dx < 0.$$

This gives sign $\lambda_{k,3n}^{(1)} = (-1)^{k-n+1}$ for $n+1 \leq k \leq 2n$. In a similar way one finds the sign of $\lambda_{k,3n}^{(1)}$ for $2n+1 \leq k \leq 3n$ by using Lemma 5.3. □

For the quadrature weights $\lambda_{k,3n}^{(2)}$ one has a similar result, which we state without proof.

THEOREM 5.5. *Let $\hat{c}$ be sufficiently large. For the quadrature weights of the second integral* (5.2) *one has*

$$\lambda_{k,3n}^{(2)} > 0, \qquad n+1 \leq k \leq 2n,$$

and

$$\text{sign } \lambda_{k,3n}^{(2)} = \begin{cases} (-1)^{k-n}, & 1 \leq k \leq n, \\ (-1)^{k+1}, & 2n+1 \leq k \leq 3n. \end{cases}$$

Observe that the quadrature weights for the nodes outside $[-d,d]$ are alternating, but the weights for the nodes closest to $[-d,d]$ are positive.

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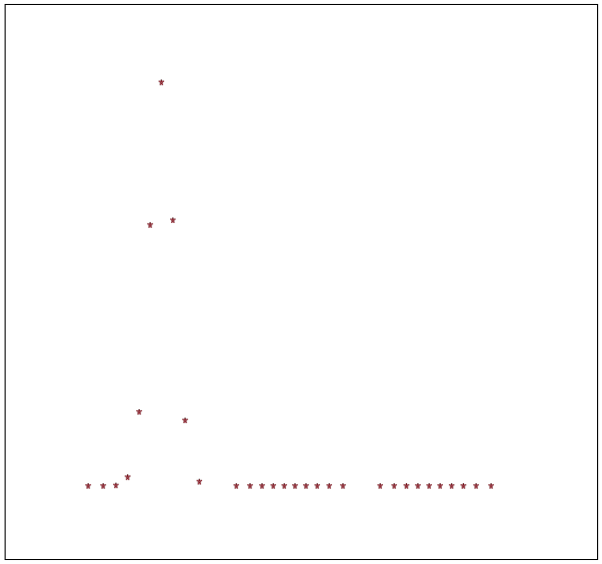


FIG. 5.1. *The quadrature weights* $\lambda_{k,30}^{(1)}$ *for the first integral* ($\hat{c} = 4.7434$).

For the quadrature nodes $\lambda_{k,3n}^{(3)}$ one has the following result:

THEOREM 5.6. *Let* $\hat{c}$ *be sufficiently large. For the quadrature weights of the third integral* (5.3) *one has*

$$\lambda_{k,3n}^{(3)} > 0, \qquad 2n+1 \leq k \leq 3n,$$

and

$$\text{sign } \lambda_{k,3n}^{(3)} = (-1)^k, \qquad 1 \leq k \leq 2n.$$

Having positive quadrature weights is a nice property, as is well known for Gaussian quadrature. The alternating quadrature weights are not so nice, but we can show that they are exponentially small.

THEOREM 5.7. *Suppose* $\hat{c}$ *is sufficiently large (see the footnote in Theorem 5.4). For the positive quadrature weights one has*

$$\limsup_{n\to\infty} \left(\lambda_{k,3n}^{(1)}\right)^{1/n} \leq e^{-V_1(x)}, \tag{5.4}$$

whenever $x_k \to x \in (-b,-a)$. *For the quadrature weights with alternating sign, it holds that*

$$\limsup_{n\to\infty} |\lambda_{k,3n}^{(1)}|^{1/n} \leq \exp\left(2U(x;\nu_1) + U(x;\nu_2) + U(x;\nu_3) - \ell_1\right) \tag{5.5}$$

whenever $x_{k,3n} \to x \in (-d,d) \cup (a,b)$.

Proof. Let $x \in (-b,-a)$. We use Lemma 5.1 to see that $\lambda_{k,3n}^{(1)} q_n(x_k) r_n(x_k) = \lambda_{k,n}$, where $\lambda_{k,n}$ are the Gaussian quadrature weights for the weight function $q_n(x) r_n(x) e^{-nV_1(x)}$. We can use the Chebyshev-Markov-Stieltjes inequalities [12, Section 3.41] for the Gaussian quadrature weights to find

$$\lambda_{k,3n}^{(1)} q_n(x_k) r_n(x_k) \leq \int_{x_{k-1}}^{x_{k+1}} q_n(x) r_n(x) e^{-nV_1(x)}\, dx.$$

By the mean value theorem, we have

$$\int_{x_{k-1}}^{x_{k+1}} q_n(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx = (x_{k+1}-x_{k-1})q_n(\xi_n)r_n(\xi_n)e^{-nV_1(\xi_n)},$$

for some $\xi_n \in (x_{k-1}, x_{k+1})$. Then, since $x_{k+1}-x_{k-1} \leq b-a$, we find

$$\limsup_{n\to\infty}\left(\lambda_{k,3n}^{(1)}\right)^{1/n} \leq e^{-V_1(x)}$$

whenever $x_k \to x \in (-b,-a)$ since

$$\lim_{n\to\infty}|q_n(x_k)|^{1/n} = \exp\bigl(-U(x;\nu_2)\bigr) = \lim_{n\to\infty}|q_n(\xi_n)|^{1/n},$$

and

$$\lim_{n\to\infty}|r_n(x_k)|^{1/n} = \exp\bigl(-U(x;\nu_3)\bigr) = \lim_{n\to\infty}|r_n(\xi_n)|^{1/n}.$$

Let $x \in (-d,d)$. We use Lemma 5.2 to find

$$|\lambda_{k,3n}^{(1)}| = \frac{1}{p_n^2(x_k)|r_n(x_k)||q_n'(x_k)|}\left|\int_{-\infty}^{\infty}\frac{q_n(x)}{x-x_k}p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx\right|.$$

For the polynomials p_n and r_n one has

$$\lim_{n\to\infty}|p_n(x)|^{1/n} = \exp\bigl(-U(x;\nu_1)\bigr), \qquad \lim_{n\to\infty}|r_n(x)| = \exp\bigl(-U(x;\nu_3)\bigr)$$

uniformly in $x \in [-d,d]$, which already gives

$$\lim_{n\to\infty}\frac{1}{p_n^2(x_k)|r_n(x_k)|} = \exp\bigl(2U(x;\nu_1)+U(x;\mu_3)\bigr).$$

For the integral we use the infinite-finite range inequality (see Proposition 3.1) to find

$$\left|\int_{-\infty}^{\infty}\frac{q_n(x)}{x-x_k}p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx\right| \leq 2\int_{-I_1}\frac{|q_n(x)|}{|x-x_k|}p_n^2(x)|r_n(x)|e^{-n(x^2+\hat{c}x)}\,dx.$$

For $\hat{c}$ sufficiently large the intervals I_1, $[-d,d]$, and $[a,b]$ are disjoint, hence for $x \in I_1$ and $x_k \in [-d,d]$ we have $|x-x_k| > \text{dist}(I_1,[-d,d]) = \delta_1$. We thus obtain

$$\left|\int_{-\infty}^{\infty}\frac{q_n(x)}{x-x_k}p_n^2(x)r_n(x)e^{-n(x^2+\hat{c}x)}\,dx\right| \leq \frac{2}{\delta_1}\int_{I_1}|q_n(x)|p_n^2(x)|r_n(x)|e^{-n(x^2+\hat{c}x)}\,dx.$$

Observe that the integrand is

$$p_n^2(x)|q_n(x)r_n(x)|e^{-n(x^2+\hat{c}x)} = \exp\Bigl(-n\bigl(2U(x;\nu_1)+U(x;\nu_2)+U(x;\nu_3)+V_1(x)\bigr)\Bigr),$$

and as $n\to\infty$, the nth root thus converges to $\exp(-\ell_1)$ when $x \in [-b,-a]$ or to a value less than or equal to $\exp(-\ell_1)$ when $x \notin [-b,-a]$. We thus have (see the third Corollary [10, p. 199] for an Angelesco system)

$$\limsup_{n\to\infty}\left(\frac{2}{\delta_1}\int_{I_1}|q_n(x)|p_n^2(x)|r_n(x)|e^{-n(x^2+\hat{c}x)}\,dx\right)^{1/n} \leq e^{-\ell_1}.$$

The behavior of the nth root of $|q_n'(x_k)|$ is more difficult because we evaluate q_n' at a point in $(-d,d)$, which is in the support of ν_2 where the zeros of q_n are dense. Clearly q_n' has $n-1$ zeros between the zeros of q_n, and the asymptotic distribution of the zeros of q_n' is the same as that of q_n, hence $|q_n'(x)|^{1/n}$ converges to $\exp\bigl(-U(x;\nu_2)\bigr)$ whenever $x \notin [-d,d]$. When $x_k \to x \in (-d,d)$ one can use the principle of descent [11, Theorem 6.8 in Chapter I] to find

$$\limsup_{n\to\infty} |q_n'(x_k)|^{1/n} \le \exp\bigl(-U(x;\nu_2)\bigr), \qquad x \in (-d,d). \tag{5.6}$$

To prove the inequality in the other direction, we look at the quadrature weights $\lambda_{k,3n}^{(2)}$ for the second integral (5.2) corresponding to the nodes on $[-d,d]$ (the zeros of q_n). These nodes are positive and related to the Gaussian quadrature nodes for the orthogonal polynomials with the weight function $p_n(x)r_n(x)e^{-nx^2}$; see Theorem 5.5. The result corresponding to (5.4) is

$$\limsup_{n\to\infty} \left(\lambda_{k,3n}^{(2)}\right)^{1/n} \le e^{-V_2(x)}.$$

On the other hand, by taking $f(x) = p_n(x)q_n^2(x)r_n(x)/(x-x_k)^2$ in (5.2), we see that the quadrature weight $\lambda_{k,3n}^{(2)}$ satisfies

$$\lambda_{k,3n}^{(2)} p_n(x_k)r_n(x_k)[q_n'(x_k)]^2 = \int_{-\infty}^{\infty} \frac{p_n(x)q_n^2(x)r_n(x)}{(x-x_k)^2} e^{-nx^2}\,dx.$$

Observe that the sign of $r_n(x)$ on $[-d,d]$ is $(-1)^n$. Hence by the infinite-finite range inequalities, one finds

$$(-1)^n \lambda_{k,3n}^{(2)} p_n(x_k)r_n(x_k)[q_n'(x_k)]^2 = (1+r_n)\int_{I_2} \frac{p_n(x)q_n^2(x)r_n(x)}{(x-x_k)^2} e^{-nx^2}\,dx,$$

where $|r_n| < 1$. On I_2 we have that $|x-x_k| \le \delta_2$, where δ_2 is the length of I_2, hence

$$(-1)^n \lambda_{k,3n}^{(2)} p_n(x_k)r_n(x_k)[q_n'(x_k)]^2 \ge \frac{1+r_n}{\delta_2^2}\int_{I_2} |p_n(x)|q_n^2(x)|r_n(x)|e^{-nx^2}\,dx,$$

from which we find

$$|q_n'(x_k)|^2 \ge \frac{1+r_n}{\delta_2^2}\frac{1}{\lambda_{k,3n}^{(2)}|p_n(x_k)r_n(x_k)|}\int_{I_2} |p_n(x)|q_n^2(x)|r_n(x)|e^{-nx^2}\,dx.$$

By taking the nth root and by using the same reasoning as before, we thus find

$$\liminf_{n\to\infty} |q_n'(x_k)|^{2/n} \ge \exp\bigl(V_2(x) + U(x;\nu_1) + U(x;\nu_3) - \ell_2\bigr).$$

Since $x \in (-d,d)$, it follows from (4.3) that the right-hand side is $\exp\bigl(-2U(x;\nu_2)\bigr)$. Combined with (5.6) we then have

$$\lim_{n\to\infty} |q_n'(x_k)|^{1/n} = e^{-U(x;\nu_2)}$$

whenever $x_k \to x \in (-d,d)$. Combining all these results gives (5.5) for $x_{k,3n} \to x \in (-d,d)$. The proof for $x_{k,3n} \to x \in (a,b)$ is obtained similarly using Lemma 5.3. ∎

The results corresponding to the quadrature weights for the second integral (5.2) and the third integral (5.3) are as follows:

THEOREM 5.8. *Suppose $\hat{c}$ is sufficiently large (see the footnote of Theorem 5.4). For the positive quadrature weights one has*

$$\limsup_{n\to\infty} \left(\lambda^{(2)}_{k,3n}\right)^{1/n} \leq e^{-V_2(x)}$$

whenever $x_k \to x \in (-d,d)$. For the quadrature weights with alternating sign, it holds that

$$\limsup_{n\to\infty} |\lambda^{(2)}_{k,3n}|^{1/n} \leq \exp\left(U(x;\nu_1) + 2U(x;\nu_2) + U(x;\nu_3) - \ell_2\right)$$

whenever $x_{k,3n} \to x \in (-b,-a) \cup (a,b)$.

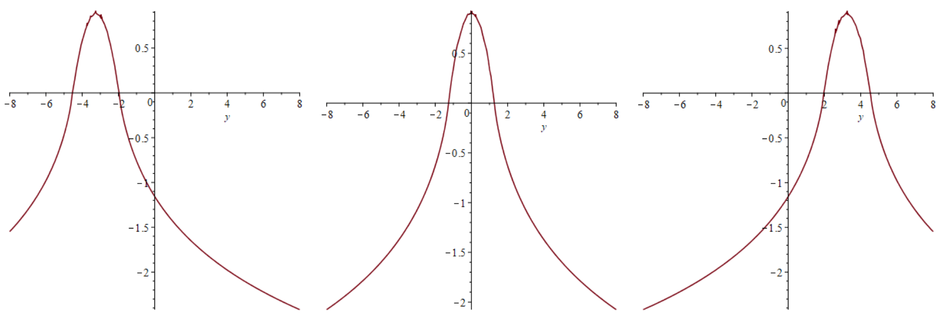
THEOREM 5.9. *Suppose $\hat{c}$ is sufficiently large (see the footnote of Theorem 5.4). For the positive quadrature weights one has*

$$\limsup_{n\to\infty} \left(\lambda^{(3)}_{k,3n}\right)^{1/n} \leq e^{-V_3(x)}$$

whenever $x_k \to x \in (a,b)$. For the quadrature weights with alternating sign, it holds that

$$\limsup_{n\to\infty} |\lambda^{(3)}_{k,3n}|^{1/n} \leq \exp\left(U(x;\nu_1) + U(x;\nu_2) + 2U(x;\nu_3) - \ell_3\right)$$

whenever $x_{k,3n} \to x \in (-b,-a) \cup (-d,d)$.



(a) $U\left(y;\nu_1^{(100)}\right)$ (b) $U\left(y;\nu_2^{(100)}\right)$ (c) $U\left(y;\nu_3^{(100)}\right)$

FIG. 5.2. *The potentials $U(x;\nu_1)$, $U(x;\nu_2)$, $U(x;\nu_3)$ approximated by using the zeros of $H_{100,100,100}$.*

All the upper bounds in Theorems 5.7–5.9 depend on the logarithmic potentials $U(x;\nu_1)$, $U(x;\nu_2), U(x;\nu_3)$, and in particular on the linear combination of them that appears in the variational conditions (4.1)–(4.6). Observe that by combining (4.3) with (5.5) we find

$$\limsup_{n\to\infty} |\lambda^{(1)}_{k,3n}|^{1/n} \leq e^{-V_2(x)} \exp\bigl(\ell_2 - U(x;\nu_2) - \ell_1 + U(x;\nu_1)\bigr)$$

whenever $x_k \to x \in (-d,d)$, and by using (4.5) we find

$$\limsup_{n\to\infty} |\lambda^{(1)}_{k,3n}|^{1/n} \leq e^{-V_3(x)} \exp\bigl(\ell_3 - U(x;\nu_3) - \ell_1 + U(x;\nu_1)\bigr)$$

whenever $x_k \to x \in (a,b)$. Hence, on $(-d,d)$ the quadrature weights are bounded from above by $e^{-V_2(x)}$ times a factor which is small since $\ell_2 - U(x;\nu_2) - \ell_1 + U(x;\nu_1) < 0$ on $[-d,d]$. On (a,b) the quadrature weights are bounded by $e^{-V_3(x)}$ times an even smaller factor since $\ell_3 = \ell_1$ (by symmetry) and $U(x;\nu_3) > U(x;\nu_2) > U(x;\nu_1)$ for $x \in (a,b)$; see Figure 5.2. This makes the alternating quadrature weights exponentially small.

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TABLE 6.1
The quadrature weights $\lambda_{k,30}^{(1)}$ *for the first integral* ($\hat{c} = 4.7434$).

k	$\lambda_{k,30}^{(1)}$	k	$\lambda_{k,30}^{(1)}$
1	$6.887653865\ 10^{-9}$	16	$-5.203778435\ 10^{-7}$
2	$4.384111578\ 10^{-6}$	17	$1.650403141\ 10^{-7}$
3	$0.359198303\ 10^{-3}$	18	$-3.822820686\ 10^{-8}$
4	$0.814961761\ 10^{-2}$	19	$5.890634594\ 10^{-9}$
5	$0.683650066\ 10^{-1}$	20	$-4.840551012\ 10^{-10}$
6	0.2410330694	21	$1.105332527\ 10^{-11}$
7	0.3725933960	22	$-7.562667367\ 10^{-12}$
8	0.2452710131	23	$3.793214538\ 10^{-12}$
9	$0.604113561\ 10^{-1}$	24	$-1.400104912\ 10^{-12}$
10	$0.380909885\ 10^{-2}$	25	$3.767415857\ 10^{-13}$
11	$6.755525278\ 10^{-6}$	26	$-7.193039657\ 10^{-14}$
12	$-5.189883715\ 10^{-6}$	27	$9.260146442\ 10^{-15}$
13	$3.848392520\ 10^{-6}$	28	$-7.331498520\ 10^{-16}$
14	$-2.434636570\ 10^{-6}$	29	$2.977117925\ 10^{-17}$
15	$1.261797315\ 10^{-6}$	30	$-3.903292274\ 10^{-19}$

6. Numerical example. In Table 6.1 and Figure 5.1 we give the quadrature weights $\lambda_{k,3n}^{(1)}$ for the zeros of $H_{10,10,10}$ with $c = 15$, which after scaling by $\sqrt{10}$ corresponds to $\hat{c} = 4.7434$. This clearly shows that the first 10 weights are positive and the remaining 20 weights are alternating in sign and very small in absolute value. The zeros and the quadrature weights behave in a similar way as for an Angelesco system (see [8]) when $\hat{c}$ is sufficiently large. Our scaling and the use of the weight functions

$$w_1(x) = e^{-n(x^2+\hat{c}x)}, \qquad w_2(x) = e^{-nx^2}, \qquad w_3(x) = e^{-n(x^2-\hat{c}x)},$$

means that we are using the densities of normal distributions with means $-\hat{c}/2, 0, \hat{c}/2$ and variance $\sigma^2 = 1/2n$. In such case we can ignore the alternating weights and only use the positive quadrature weights $\{\lambda_{k,3n}^{(1)} : 1 \leq k \leq n\}$ to approximate the first integral (5.1). In a similar way, when we approximate the second integral (5.2) we can ignore the alternating weights and only use the positive weights $\{\lambda_{k,3n}^{(2)} : n+1 \leq k \leq 2n\}$, and for approximating the third integral, one can only use $\{\lambda_{k,3n}^{(3)} : 2n+1 \leq k \leq 3n\}$.

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