Electronic Transactions on Numerical Analysis.
Volume 58, pp. 568–581, 2023.

ISSN 1068–9613.
DOI: 10.1553/etna_vol58s568

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AN EVOLUTIONARY APPROACH TO THE COEFFICIENT PROBLEMS IN THE CLASS OF STARLIKE FUNCTIONS*

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Abstract. In this paper, we apply the differential evolution algorithm as a new approach to solve some coefficient problems within Geometric Function Theory. We find sharp bounds for the determinant of the Hankel matrix $H_{4,1}(f)$ and the determinants of all its sub-matrices for the class of starlike functions, i.e., for the class of all analytic injective functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$ such that $f(\mathbb{D})$ is a starlike set with respect to the origin. In addition, a relevant conjecture regarding some coefficient functionals related to the Zalcman functional is formulated.

Key words. differential evolution algorithm, Hankel determinant, starlike function, Carathéodory class and Zalcman functional

AMS subject classifications. 65K05, 30C45, 30C50

1. Introduction. The differential evolution algorithm is a very powerful and effective numerical computation technique used in many branches of mathematics as well as in applied sciences. In this paper we illustrate the application of this algorithm to coefficient problems in Geometric Function Theory (GFT). A solution to the open problem of finding sharp estimates for the fourth-degree Hankel determinant in one of the most important classes in GFT, namely the class of starlike functions, is presented. As we will further justify, the differential evolution method can be treated as an auxiliary tool for solving coefficient problems within GFT.

The fundamental subclasses of the class $\mathrm{Hol}(\mathbb{D})$ of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ that underlies Geometric Function Theory (GFT) is the class $\mathcal{A}$ of all analytic normalized functions f, i.e., functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 := 1,\ z \in \mathbb{D}, \tag{1.1}$$

and its subclass $\mathcal{S}$ of univalent (analytic and injective) functions. Apart from the class $\mathcal{S}$ itself, its subclasses, distinguished by established geometrical properties of plane sets, are of fundamental importance in the theory. Historically they are: the class of convex functions $\mathcal{S}^c$ introduced by Study [42] in 1913, the class of starlike functions $\mathcal{S}^*$ defined by Alexander [1], the class of spirallike functions introduced by Špaček [40], the class of functions convex in the direction of the imaginary axis defined by Robertson [37] in 1936, and others (cf. [13, 16]). Since this paper deals with starlike functions, we recall their definition. A function $f : \mathbb{D} \to \mathbb{C}$ is called starlike with respect to the origin, shortly starlike, if $f \in \mathcal{S}$ and $f(\mathbb{D})$ is a starlike set, i.e., the line segment $[0, w] := \{tw : 0 \le t \le 1\}$ lies in $f(\mathbb{D})$ for every $w \in f(\mathbb{D})$. The family of such functions is denoted by $\mathcal{S}^*$.

Due to the representation (1.1) of functions from the class $\mathcal{S}$ and thus also from each of its subclasses, issues related to the properties of their coefficients are crucial to describe the analytical properties of the studied classes. For $k \in \mathbb{Z}$, let $\mathbb{Z}_k := \{n \in \mathbb{Z} : n \ge k\}$. Particularly, $\mathbb{N} = \mathbb{Z}_1$. In 1916, Bieberbach [4] formulated the famous conjecture, namely that the sharp inequality $|a_n| \le n$ holds for every $n \in \mathbb{Z}_2$ and every function in the class $\mathcal{S}$ and

*Received June 8, 2023. Accepted September 13, 2023. Published online on November 29, 2023. Recommended by Tom DeLillo.
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that equality holds for the Koebe function $K \in \mathcal{S}$ defined as

$$K(z) := \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D}, \tag{1.2}$$

and its rotations. Bieberbach confirmed that his conjecture is true for the second coefficient. The search for a proof of the Bieberbach conjecture resulted in the development of many advanced research techniques such as the development of variational methods in complex analysis or the creation of the Loewner chain theory [31]. The final proof of this conjecture was given by de Branges in 1985 [12].

Regardless of the whole class $\mathcal{S}$, the problem of estimating the coefficients was transferred to the aforementioned subclasses in $\mathcal{S}$. For starlike functions, which are the subject of this paper, the Bieberbach problem was solved by Nevanlinna [33] already in 1921, who proved that $|a_n| \le n$ holds for every $n \in \mathbb{Z}_2$ and every function in the class $\mathcal{S}^*$, with the Koebe function (1.2) and its rotations being extremal. However, the same estimate for the coefficients, both in the class of starlike functions $\mathcal{S}^*$, which is a proper subclass of the class $\mathcal{S}$, and in the whole class $\mathcal{S}$, with the same extremal function, does not distinguish between the class $\mathcal{S}$ and its subclass $\mathcal{S}^*$. Therefore, it is natural to consider more complicated coefficient functionals giving a deeper knowledge of the class $\mathcal{S}$ and its subclasses. Historically, one of the first is the Fekete-Szegő functional introduced in 1933 [15], i.e., given $\lambda \in [0,1]$, let $\mathcal{S} \ni f \mapsto \Phi_\lambda(f) := a_3 - \lambda a_2^2$. Only in the case of $\lambda = 0$ and $\lambda = 1$, the upper bound for $|\Phi_\lambda(f)|$ is the same in both classes $\mathcal{S}$ and $\mathcal{S}^*$ (see, e.g., [19] for further references).

In the 60s, coefficient functionals were first considered as determinants of the Hankel matrices $H_{q,n}(f)$ defined as follows: for $q, n \in \mathbb{N}$ and $f \in \mathcal{A}$ of the form (1.1), let

$$H_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{bmatrix}.$$

General results for Hankel determinants $H_{q,n}(f)$ with applications can be found, e.g., in [7, 34, 35, 39].

Hankel matrices have remarkable applications in many areas of mathematics as well as in applied sciences. One such example is the method of moments in statistics, one of the methods for the estimation of population parameters. The method of moments applied to polynomial distributions produces a system of equations, whose solution involves the inversion of a Hankel matrix, i.e., $\boldsymbol{\theta} = H^{-1}\mathbf{E}$, where $\boldsymbol{\theta}$ is a vector of unknown weights (coefficients of a polynomial distributions), $\mathbf{E}$ a vector of sample moments, and H is a Hankel matrix. Hankel matrices appear in the theory of Markov processes, which have important applications in physics, chemistry, economics, bioinformatics, signal processing, information theory, and many others. Let us also mention that the Hamburger moment problem has a solution if and only if the Hankel kernel, which is an infinite Hankel matrix H with entries being nonnegative integers $(m_0, m_1, \dots)$, is positive definite; as a conclusion this means that the entries m_k are statistical moments. Among other things, for these reasons, the study of Hankel matrices in GFT is an important and at the same time a difficult problem. The estimation of the Hankel determinant of $H_{q,n}(f)$, particularly of the determinant of $H_{q,1}(f)$, for subclasses in the class $\mathcal{S}$ has been developed intensively in the last 20 years (e.g., [8, 9] and [36] with further references).

In the early 70s, Zalcman considered the functional $\mathcal{S} \ni f \mapsto J_n(f) := a_{2n-1} - a_n^2$, for $n \in \mathbb{Z}_2$, over the class $\mathcal{S}$. He posed the famous conjecture that if $f \in \mathcal{S}$ and is given by (1.1), then $|J_n(f)| \leq (n-1)^2$ for $n \in \mathbb{Z}_2$, with equality for the Koebe function and its rotations. This conjecture was confirmed by Krushkal for $n = 3$ [23] and for $n = 4, 5, 6$ [24]. The case $n = 2$ was shown by Bieberbach [4] (e.g., [16, Vol. I, p. 35]). Zalcman's conjecture remains open for $n \in \mathbb{Z}_7$. For the class of starlike functions, the Zalcman conjecture has been proved in [5].

In 1999 Ma [32] generalized the Zalcman functional as follows:

$$\mathcal{S} \ni f \mapsto J_{m,n}(f) := a_{m+n-1} - a_n a_m, \qquad \text{for } m, n \in \mathbb{Z}_2.$$

He conjectured that if $f \in \mathcal{S}$, then for $m, n \in \mathbb{Z}_2$,

$$|J_{m,n}(f)| \leq (m-1)(n-1),$$

with equality for the Koebe function and its rotations, and he confirmed the conjecture for functions in the class $\mathcal{S}^*$ and the subclass $\mathcal{S}_{\mathbb{R}}$ of $\mathcal{S}$ of functions having real coefficients. For further results and references on the generalized Zalcman functional, see, e.g., [10, 14].

We now note that both the Zalcman functional J_n and the generalized Zalcman functional $J_{m,n}$ are determinants of sub-matrices of every Hankel matrix $H_{q,1}(f)$ with $q \geq \max\{m, n\}$. Namely,

$$J_n(f) = \begin{vmatrix} a_1 & a_n \\ a_n & a_{2n-1} \end{vmatrix}$$

and

$$J_{m,n}(f) = \begin{vmatrix} a_1 & a_n \\ a_m & a_{m+n-1} \end{vmatrix}.$$

For this reason it is natural to introduce the following functional: for $k \in \mathbb{N}$, $m, n \in \mathbb{Z}_{k+1}$, let

$$\mathcal{A} \ni f \mapsto J_{m,n,k}(f) := a_k a_{m+n-k} - a_n a_m = \begin{vmatrix} a_k & a_n \\ a_m & a_{m+n-k} \end{vmatrix}.$$

Due to the results of Section 3, we formulate the following conjecture:

$$|J_{m,n,k}(f)| \leq (m-k)(n-k), \quad f \in \mathcal{S}^*,$$

with equality for the Koebe function and its rotation. We think that this conjecture is meaningful and true for the whole class $\mathcal{S}$.

The problem of finding sharp estimates for the Hankel determinants is in general technically difficult. The applied computing methods are able to find such estimates for the determinant $\det H_{2,2}(f) = a_2 a_4 - a_3^2$ on many known subclasses in the class $\mathcal{A}$, in particular in the subclasses of $\mathcal{S}$. It is much more difficult to find sharp bounds for the third-order determinant $\det H_{3,1}(f)$ in known classes of analytic functions. Such a sharp estimate was obtained for the class of convex functions $\mathcal{S}^c$ in [21], where it was shown that $|\det H_{3,1}(f)| \leq 4/135$; for the class of starlike functions of order $1/2$ in [29]; it was shown that $|\det H_{3,1}(f)| \leq 1/9$ for the classes $\mathcal{T}(0)$ and $\mathcal{T}(1/2)$ in [20]. Here, given $\alpha \in [0, 1)$, the class $\mathcal{T}(\alpha)$ consists of all $f \in \mathcal{A}$ such that $\operatorname{Re} f(z)/z > \alpha$ for $z \in \mathbb{D}$. For the class of starlike functions $\mathcal{S}^*$ being of interest in this paper, the sharp inequality $|\det H_{3,1}(f)| \leq 4/9$ was shown in a recent paper [22].

Taking into account the methods used to estimate the Hankel determinants of second and third-order, the problem of finding a sharp estimate for the Hankel determinant $\det H_{4,1}(f)$ in selected subclasses of $\mathcal{S}$ is extremely difficult. It is not a problem to find some estimate, but

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the difficulty is to arrive at a sharp one. The authors do not know any such sharp result for determinants of $H_{4,1}(f)$ for basic subclasses of univalent functions. Since both the Zalcman functional and the generalized Zalcman functional are defined by sub-matrices in $H_{q,1}(f)$, it is natural to consider all square sub-matrices of $H_{q,1}(f)$.

For this reason, the main goal of this paper is to propose the evolutionary approach to estimate the determinants of such matrices as a fresh computational idea in GFT. We estimate the determinant of the Hankel matrix $H_{4,1}(f)$ and determinants of all its square sub-matrices in the class $\mathcal{S}^*$. Taking into account the obtained upper bounds by applying the evolutionary approach, a relevant general conjecture about the functional $J_{m,n,k}$ defined here is formulated.

At the end let us emphasize that the differential evolution method can be treated as an auxiliary tool for analyzing similar problems within GFT.

2. Preliminaries. The class $\mathcal{S}^*$ of starlike functions studied in this paper was introduced by Alexander in 1915 [1], where he also formulated their analytic characterization, which in detail was elaborated by Nevanlinna [33] (e.g., [13, p. 41]). A different concept of deriving the analytic formula for starlike functions was demonstrated in [27, 28]. In [30] the authors provide an alternative, unified, and self-contained proof of the theorem below due to Alexander [1], which, by the way, may be adopted in more general geometrical concepts.

THEOREM 2.1. *For every $f \in \mathcal{A}$, the following equivalence holds: $f \in \mathcal{S}^*$ if and only if*

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \qquad z \in \mathbb{D}\setminus\{0\}. \tag{2.1}$$

Denote by $\mathcal{P}$ the subclass of $\mathrm{Hol}(\mathbb{D})$ of all analytic functions p having a positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in \mathbb{D}. \tag{2.2}$$

It is well known ([6], cf. [16, Vol. I, p. 80]) that for every $p \in \mathcal{P}$ of the form (2.2), the following sharp estimate holds true:

$$|c_n| \le 2, \qquad n \in \mathbb{N}. \tag{2.3}$$

Now we define the m-th coefficient region $C_m(\mathcal{P})$.

DEFINITION 2.2. *For $m \in \mathbb{N}$, let*

$$C_m(\mathcal{P}) := \left\{(c_1,\ldots,c_m) \in \mathbb{C}^m : \exists_{p\in\mathcal{P}}\, c_k = p^{(k)}(0)/k!,\ k=1,\ldots,m\right\}.$$

Given $r > 0$, let $\overline{\mathbb{D}}_r := \{z \in \mathbb{C} : |z| \le r\}$. For every $m \in \mathbb{N}$, $C_m(\mathcal{P})$ is a compact convex set in $\mathbb{C}^m$ (e.g., [17, p. 162, Corollary 9.11]). From (2.3) it follows also that $C_m(\mathcal{P}) \subset \overline{\mathbb{D}}_2^{\,m}$.

Let $\mathbf{H} := (\mathrm{Hol}(\mathbb{D}), \mathcal{T})$ be the topological space where $\mathcal{T}$ is the topology of uniform convergence on compact subsets of $\mathbb{D}$. Let $\mathcal{F}$ be a compact subset of $\mathrm{Hol}(\mathbb{D})$ in the space $\mathbf{H}$. Recall that a function $f \in \mathrm{Hol}(\mathbb{D})$ is called the support point of $\mathcal{F}$ if $f \in \mathcal{F}$ and there exists a linear functional Λ on $\mathrm{Hol}(\mathbb{D})$ such that $\mathrm{Re}\,\Lambda$ is non-constant on $\mathcal{F}$ and

$$\mathrm{Re}\,\Lambda(f) = \max\left\{\mathrm{Re}\,\Lambda(g) : g \in \mathcal{F}\right\}.$$

The set of all support points of $\mathcal{F}$ is denoted as $\mathrm{supp}\,\mathcal{F}$.

Given $m \in \mathbb{N}$, let $\mathcal{P}_m$ denote the set of all functions of the form

$$p(z) = \sum_{j=1}^{m} \lambda_j L(x_j z), \qquad z \in \mathbb{D}, \tag{2.4}$$

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where the x_j are distinct points of $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, the parameters $\lambda_j \geq 0$, for $j = 1, \ldots, m$, are such that

$$\sum_{j=1}^{m} \lambda_j = 1, \tag{2.5}$$

and $L \in \mathcal{P}$ is defined as

$$L(z) := \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n, \qquad z \in \mathbb{D}. \tag{2.6}$$

Clearly, $\mathcal{P}_m \subset \mathcal{P}$ for all $m \in \mathbb{N}$. It is well known (e.g., [17, p. 94]) that

$$\operatorname{supp} \mathcal{P} = \bigcup_{m \in \mathbb{N}} \mathcal{P}_m.$$

The theorem below, which can be found in [38, Theorem C] is the basis for further considerations. As the author remarked, this theorem is equivalent to Theorem B shown by Hummel in [18].

THEOREM 2.3. *Let $m \in \mathbb{N}$ and $\Phi : C_m(\mathcal{P}) \to \mathbb{C}$ be a continuous function, analytic in the interior of $C_m(\mathcal{P})$. If a function $p \in \mathcal{P}$ maximizes $\operatorname{Re} \Phi$ on $C_m(\mathcal{P})$, then p is of the form* (2.4)*, i.e., $p \in \mathcal{P}_m$.*

From (2.4) and (2.6), the next lemma follows:

LEMMA 2.4. *Let $m \in \mathbb{N}$. For $p \in \mathcal{P}_m$ of the form* (2.2)*, it holds that*

$$c_n = 2\sum_{j=1}^{m} \lambda_j x_j^n, \qquad n \in \mathbb{N}, \tag{2.7}$$

where the x_j are distinct points of $\mathbb{T}$ and $\lambda_j \geq 0$, for $j = 1, \ldots, m$, satisfy (2.5).

3. Differential evolution. Evolutionary algorithms are among the best general methods for optimization. Differential evolution is based on population evolution. It has four components:

- initialization,
- mutation,
- crossover,
- selection.

Without loss of generality, we consider minimization problems for a function $f : \mathbb{R}^D \mapsto \mathbb{R}$, $D \in \mathbb{Z}_2$, (maximization can be performed by considering the function $h = -f$ instead). The main goal of the algorithm is to find a global minimum of f on some set (for simplicity we will assume it to be a D-dimensional cube).

At the beginning we set 3 parameters: $CR \in [0, 1]$, $F \in [0, 2]$, $NP \in \mathbb{Z}_5$. The set G is a population—a set of NP randomly chosen vectors from the domain of the function f. Next, for each vector $\mathbf{x} = [x_1, \ldots, x_D]$ from G, we perform the following operations:

- we randomly choose three vectors $\mathbf{a} = [a_1, \ldots, a_D]$, $\mathbf{b} = [b_1, \ldots, b_D]$, and $\mathbf{c} = [c_1, \ldots, c_D] \in G$ different in pairs and different from $\mathbf{x}$,
- we randomly choose an integer R from the set $\{1, 2, \ldots, D\}$,
- we define the vector $\mathbf{y} = [y_1, \ldots, y_D]$ as follows:
  - we choose the number r_i at random according to the normal distribution $N(0, 1)$, for $i \in \{1, \ldots, D\}$,

- if $r_i < CR$ or $i = R$, then we set $y_i = a_i + F(b_i - c_i)$; otherwise we set $y_i = x_i$,
- if $\mathbf{y}$ belongs to the domain of function f and $f(\mathbf{y}) \leqslant f(\mathbf{x})$, then we substitute $\mathbf{y} := \mathbf{x}$.

The algorithm was introduced by Storn and Price in 1996 [41]. The choice of the parameters has a big impact on the speed of optimization. Mathematical correctness has been considered in [11, 43].

4. Application of differential evolution to starlike functions. For further consideration let us fix some notation. Let M be a square matrix of order $n \in \mathbb{Z}_2$. Given $1 \leq k < n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, by $M_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ we denote a sub-matrix of M by removing k rows and k columns numbered $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$, respectively. Since our interest is related to the matrix $H_{4,1}(f)$, to simplify the notation, set

$$M = M(f) := H_{4,1}(f).$$

For $\theta \in \mathbb{R}$ and $f \in \mathcal{A}$, let $f_\theta(z) := \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z),\ z \in \mathbb{D}$, denote a rotation of f. By (1.1) we have

$$f_\theta(z) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{\mathrm{i}(n-1)\theta} z^n, \qquad a_1 = 1,\ z \in \mathbb{D}.$$

Observe now that

$$\det M(f_\theta) = \det H_{4,1}(f_\theta) = \mathrm{e}^{12\mathrm{i}\theta} \det H_{4,1}(f).$$

Therefore the estimation of $|\det H_{4,1}(f)|$ after suitable rotation of f is equivalent to the estimation of $\mathrm{Re}(\det H_{4,1}(f))$. A similar property holds for the determinant of each square sub-matrix of $H_{4,1}(f)$. For this reason, to compute an upper bound for the modulus of the determinants of $H_{4,1}(f)$ and of its square sub-matrices, we apply Theorem 2.3.

Let $f \in \mathcal{S}^*$ be of the form (1.1). Then by (2.1) there exists a function $p \in \mathcal{P}$ of the form (2.2) such that

$$zf'(z) = p(z)f(z), \quad z \in \mathbb{D}. \tag{4.1}$$

Substituting the series (1.1) and (2.2) into (4.1) and by comparing the corresponding coefficients we obtain

$$na_n = a_n + \sum_{k=1}^{n-1} c_{n-k} a_k, \qquad n \in \mathbb{N},\ a_1 = 1.$$

Hence we get

$$\begin{aligned}
a_2 &= c_1,\\
a_3 &= \frac{c_1^2}{2} + \frac{c_2}{2},\\
a_4 &= \frac{c_1^3}{6} + \frac{c_1 c_2}{2} + \frac{c_3}{3},\\
a_5 &= \frac{c_1^4}{24} + \frac{c_1^2 c_2}{4} + \frac{c_1 c_3}{3} + \frac{c_2^2}{8} + \frac{c_4}{4},\\
a_6 &= \frac{c_1^5}{120} + \frac{c_1^3 c_2}{12} + \frac{c_1^2 c_3}{6} + \frac{c_1 c_2^2}{8} + \frac{c_1 c_4}{4} + \frac{c_2 c_3}{6} + \frac{c_5}{5},\\
a_7 &= \frac{c_1^6}{720} + \frac{c_1^4 c_2}{48} + \frac{c_1^3 c_3}{18} + \frac{c_1^2 c_2^2}{16} + \frac{c_1^2 c_4}{8} + \frac{c_1 c_2 c_3}{6} + \frac{c_1 c_5}{5} + \frac{c_2^3}{48} + \frac{c_2 c_4}{8} + \frac{c_3^2}{18} + \frac{c_6}{6}.
\end{aligned}$$

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Observe that each function Φ_k, $k = 1, \ldots, 31$, below is continuous on $C_m(\mathcal{P})$ and analytic in the interior of $C_m(\mathcal{P})$ with the corresponding m. Therefore by Theorem 2.3 our computation in each case is restricted to functions p in $\mathcal{P}_m$, hence, to the functions (2.4) having coefficients c_n of the form (2.7). When applying the evolutionary algorithm, we use $x_j = \mathrm{e}^{\mathrm{i}\theta_j}$, with $\theta_j \in \mathbb{R}$ for $j = 1, \ldots, m$.

4.1. Results of the evolutionary algorithm. Here the symbol $\approx$ denotes the numerical approximation (not rounding).

(1) $$a_1 = 1$$

(2) $$a_2 = c_1 =: \Phi_1(c_1)$$
$$\max|a_2| \approx 1.9999999999635196$$

(3) $$a_3 = \frac{c_1^2}{2} + \frac{c_2}{2} =: \Phi_2(c_1, c_2)$$
$$\max|a_3| \approx 2.999999999966037$$

(4) $$a_4 = \frac{c_1^3}{6} + \frac{c_1 c_2}{2} + \frac{c_3}{3} =: \Phi_3(c_1, c_2, c_3)$$
$$\max|a_4| \approx 3.9999999999802265$$

(5) $$a_5 = \frac{c_1^4}{24} + \frac{c_1^2 c_2}{4} + \frac{c_1 c_3}{3} + \frac{c_2^2}{8} + \frac{c_4}{4} =: \Phi_4(c_1, c_2, c_3, c_4)$$
$$\max|a_5| \approx 4.9999999999790745$$

(6) $$a_6 = \frac{c_1^5}{120} + \frac{c_1^3 c_2}{12} + \frac{c_1^2 c_3}{6} + \frac{c_1 c_2^2}{8} + \frac{c_1 c_4}{4} + \frac{c_2 c_3}{6} + \frac{c_5}{5} =: \Phi_5(c_1, c_2, c_3, c_4, c_5)$$
$$\max|a_6| \approx 5.999999999994299$$

(7) $$a_7 = \frac{c_1^6}{720} + \frac{c_1^4 c_2}{48} + \frac{c_1^3 c_3}{18} + \frac{c_1^2 c_2^2}{16} + \frac{c_1^2 c_4}{8} + \frac{c_1 c_2 c_3}{6} + \frac{c_1 c_5}{5} + \frac{c_2^3}{48} + \frac{c_2 c_4}{8} + \frac{c_3^2}{18} + \frac{c_6}{6} =: \Phi_6(c_1, c_2, c_3, c_4, c_5, c_6)$$
$$\max|a_7| \approx 6.9999999996337685$$

(8) $$\det M_{3,4}^{3,4} = \det H_{2,1}(f) = J_2(f) = a_3 - a_2^2 = -\frac{c_1^2}{2} + \frac{c_2}{2} =: \Phi_7(c_1, c_2)$$
$$\max\left|\det M_{3,4}^{3,4}\right| = \max|\det H_{2,1}(f)| \approx 0.9999999224498696$$

(9) $$\det M_{3,4}^{2,4} = J_{2,3}(f) = a_4 - a_2 a_3 = -\frac{c_1^3}{3} + \frac{c_3}{3} =: \Phi_8(c_1, c_2, c_3)$$
$$\max\left|\det M_{3,4}^{2,4}\right| \approx 1.9999999997271551$$

(10) $$\det M_{3,4}^{2,3} = J_{2,4}(f) = a_5 - a_2 a_4 = -\frac{c_1^4}{8} - \frac{c_1^2 c_2}{4} + \frac{c_2^2}{8} + \frac{c_4}{4} =: \Phi_9(c_1, c_2, c_3, c_4)$$
$$\max\left|\det M_{3,4}^{2,3}\right| \approx 2.999999999995327$$

(11) $$\det M_{2,4}^{2,4} = J_3(f) = a_5 - a_3^2 = -\frac{5c_1^4}{24} - \frac{c_1^2 c_2}{4} + \frac{c_1 c_3}{3} - \frac{c_2^2}{8} + \frac{c_4}{4} =: \Phi_{10}(c_1, c_2, c_3, c_4)$$
$$\max\left|\det M_{2,4}^{2,4}\right| \approx 3.9999999937369553$$

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$$\begin{aligned}\det M^{2,3}_{2,4} &= J_{3,4}(f) = a_6 - a_3a_4 = -\frac{3c_1^5}{40} - \frac{c_1^3c_2}{4} - \frac{c_1c_2^2}{8} + \frac{c_1c_4}{4} + \frac{c_5}{5} \\ &=: \Phi_{11}(c_1,c_2,c_3,c_4,c_5) \\ \max\left|\det M^{2,3}_{2,4}\right| &\approx 5.999999995128978\end{aligned} \tag{12}$$

$$\begin{aligned}\det M^{2,3}_{2,3} &= J_4(f) = a_7 - a_4^2 \\ &= -\frac{19c_1^6}{720} - \frac{7c_1^4c_2}{48} - \frac{c_1^3c_3}{18} - \frac{3c_1^2c_2^2}{16} + \frac{c_1^2c_4}{8} - \frac{c_1c_2c_3}{6} \\ &\quad + \frac{c_1c_5}{5} + \frac{c_2^3}{48} + \frac{c_2c_4}{8} - \frac{c_3^2}{18} + \frac{c_6}{6} \\ &=: \Phi_{12}(c_1,c_2,c_3,c_4,c_5,c_6) \\ \max\left|\det M^{2,3}_{2,3}\right| &\approx 8.999999999949889\end{aligned} \tag{13}$$

$$\begin{aligned}\det M^{1,4}_{1,4} &= H_{2,2}(f) = J_{3,3,2}(f) = a_2a_4 - a_3^2 = -\frac{c_1^4}{12} + \frac{c_1c_3}{3} - \frac{c_2^2}{4} \\ &=: \Phi_{13}(c_1,c_2,c_3) \\ \max\left|\det M^{1,4}_{1,4}\right| &\approx 0.999999913584528\end{aligned} \tag{14}$$

$$\begin{aligned}\det M^{2,4}_{1,4} &= J_{3,4,2}(f) = a_2a_5 - a_3a_4 \\ &= -\frac{c_1^5}{24} - \frac{c_1^3c_2}{12} + \frac{c_1^2c_3}{6} - \frac{c_1c_2^2}{8} + \frac{c_1c_4}{4} - \frac{c_2c_3}{6} \\ &=: \Phi_{14}(c_1,c_2,c_3,c_4) \\ \max\left|\det M^{2,4}_{1,4}\right| &\approx 1.9999999999537437\end{aligned} \tag{15}$$

$$\begin{aligned}\det M^{2,3}_{1,4} &= J_{3,5,2}(f) = a_2a_6 - a_3a_5 \\ &= -\frac{c_1^6}{80} - \frac{c_1^4c_2}{16} - \frac{c_1^2c_2^2}{16} + \frac{c_1^2c_4}{8} + \frac{c_1c_5}{5} - \frac{c_2^3}{16} - \frac{c_2c_4}{8} \\ &=: \Phi_{15}(c_1,c_2,c_3,c_4,c_5) \\ \max\left|\det M^{2,3}_{1,4}\right| &\approx 2.9999999999997455\end{aligned} \tag{16}$$

$$\begin{aligned}\det M^{2,4}_{1,3} &= J_{4,4,2}(f) = a_2a_6 - a_4^2 \\ &= -\frac{7c_1^6}{360} - \frac{c_1^4c_2}{12} + \frac{c_1^3c_3}{18} - \frac{c_1^2c_2^2}{8} + \frac{c_1^2c_4}{4} - \frac{c_1c_2c_3}{6} + \frac{c_1c_5}{5} - \frac{c_3^2}{9} \\ &=: \Phi_{16}(c_1,c_2,c_3,c_4,c_5) \\ \max\left|\det M^{2,4}_{1,3}\right| &\approx 3.9999999995765996\end{aligned} \tag{17}$$

$$\begin{aligned}\det M^{2,3}_{1,3} &= J_{4,5,2}(f) = a_2a_7 - a_4a_5 \\ &= -\frac{c_1^7}{180} - \frac{c_1^5c_2}{24} - \frac{c_1^4c_3}{72} - \frac{c_1^3c_2^2}{12} + \frac{c_1^3c_4}{12} - \frac{c_1^2c_2c_3}{12} \\ &\quad + \frac{c_1^2c_5}{5} - \frac{c_1c_2^3}{24} - \frac{c_1c_3^2}{18} + \frac{c_1c_6}{6} - \frac{c_2^2c_3}{24} - \frac{c_3c_4}{12} \\ &=: \Phi_{17}(c_1,c_2,c_3,c_4,c_5,c_6) \\ \max\left|\det M^{2,3}_{1,3}\right| &\approx 5.999999999994852\end{aligned} \tag{18}$$

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$$\begin{aligned}
\det M_{1,2}^{3,4} &= J_{4,4,3}(f) = a_3a_5 - a_4^2 \\
&= -\frac{c_1^6}{144} - \frac{c_1^4c_2}{48} + \frac{c_1^3c_3}{18} - \frac{c_1^2c_2^2}{16} + \frac{c_1^2c_4}{8} - \frac{c_1c_2c_3}{6} \\
&\quad + \frac{c_2^3}{16} + \frac{c_2c_4}{8} - \frac{c_3^2}{9} \\
&=: \Phi_{18}(c_1,c_2,c_3,c_4,c_5,c_6)
\end{aligned} \tag{19}$$

$$\max\left|\det M_{1,2}^{3,4}\right| \approx 0.9999999497375464$$

$$\begin{aligned}
\det M_{1,2}^{2,4} &= J_{4,5,3}(f) = a_3a_6 - a_4a_5 \\
&= -\frac{c_1^7}{360} - \frac{c_1^5c_2}{60} + \frac{c_1^4c_3}{72} - \frac{c_1^3c_2^2}{24} + \frac{c_1^3c_4}{12} - \frac{c_1^2c_2c_3}{12} + \frac{c_1^2c_5}{10} \\
&\quad - \frac{c_1c_3^2}{9} + \frac{c_2^2c_3}{24} + \frac{c_2c_5}{10} - \frac{c_3c_4}{12} \\
&=: \Phi_{19}(c_1,c_2,c_3,c_4,c_5)
\end{aligned} \tag{20}$$

$$\max\left|\det M_{1,2}^{2,4}\right| \approx 1.9999999999408111$$

$$\begin{aligned}
\det M_{1,2}^{2,3} &= J_{4,6,3}(f) = a_3a_7 - a_4a_6 \\
&= -\frac{c_1^8}{1440} - \frac{c_1^6c_2}{144} - \frac{c_1^5c_3}{360} - \frac{c_1^4c_2^2}{48} + \frac{c_1^4c_4}{48} - \frac{c_1^3c_2c_3}{36} \\
&\quad + \frac{c_1^3c_5}{15} - \frac{c_1^2c_2^3}{48} - \frac{c_1^2c_3^2}{36} + \frac{c_1^2c_6}{12} - \frac{c_1c_2^2c_3}{24} \\
&\quad - \frac{c_1c_3c_4}{12} + \frac{c_2^4}{96} + \frac{c_2^2c_4}{16} - \frac{c_2c_3^2}{36} + \frac{c_2c_6}{12} - \frac{c_3c_5}{15} \\
&=: \Phi_{20}(c_1,c_2,c_3,c_4,c_5,c_6)
\end{aligned} \tag{21}$$

$$\max\left|\det M_{1,2}^{2,3}\right| \approx 2.9999999997966227$$

$$\begin{aligned}
\det M_4^4 &= \det H_{3,1}(f) = -\frac{c_1^6}{144} + \frac{c_1^4c_2}{48} + \frac{c_1^3c_3}{18} - \frac{c_1^2c_2^2}{16} \\
&\quad - \frac{c_1^2c_4}{8} + \frac{c_1c_2c_3}{6} - \frac{c_2^3}{16} + \frac{c_2c_4}{8} - \frac{c_3^2}{9} \\
&=: \Phi_{21}(c_1,c_2,c_3,c_4)
\end{aligned} \tag{22}$$

$$\max\left|\det M_4^4\right| = \max\left|\det H_{3,1}(f)\right| \approx 0.44444444442415865$$

$$\begin{aligned}
\det M_4^3 &= -\frac{c_1^7}{240} + \frac{c_1^5c_2}{240} + \frac{c_1^4c_3}{24} - \frac{c_1^3c_2^2}{48} - \frac{c_1^3c_4}{24} + \frac{c_1^2c_2c_3}{12} - \frac{c_1^2c_5}{10} \\
&\quad - \frac{c_1c_2^3}{16} + \frac{c_1c_2c_4}{8} - \frac{c_2^2c_3}{24} + \frac{c_2c_5}{10} - \frac{c_3c_4}{12} \\
&=: \Phi_{22}(c_1,c_2,c_3,c_4,c_5)
\end{aligned} \tag{23}$$

$$\max\left|\det M_4^3\right| \approx 0.42427115980329017$$

$$\begin{aligned}
\det M_4^2 &= -\frac{c_1^8}{960} + \frac{c_1^5c_3}{60} - \frac{c_1^4c_2^2}{96} - \frac{c_1^3c_5}{15} + \frac{c_1^2c_2c_4}{8} \\
&\quad - \frac{c_1c_2^2c_3}{12} + \frac{c_2^4}{64} + \frac{c_3c_5}{15} - \frac{c_4^2}{16} \\
&=: \Phi_{23}(c_1,c_2,c_3,c_4,c_5)
\end{aligned} \tag{24}$$

$$\max\left|\det M_4^2\right| \approx 0.3699523786405721$$

$$\begin{aligned}\det M_1^4 &= -\frac{c_1^9}{8640} + \frac{c_1^6 c_3}{360} - \frac{c_1^5 c_2^2}{480} - \frac{c_1^4 c_5}{60} + \frac{c_1^3 c_2 c_4}{24} - \frac{c_1^2 c_2^2 c_3}{24} + \frac{c_1 c_2^4}{64} \\ &\quad + \frac{c_1 c_3 c_5}{15} - \frac{c_1 c_4^2}{16} - \frac{c_2^2 c_5}{20} + \frac{c_2 c_3 c_4}{12} - \frac{c_3^3}{27} \\ &=: \Phi_{24}(c_1, c_2, c_3, c_4, c_5)\end{aligned} \tag{25}$$

$$\max\left|\det M_1^4\right| \approx 0.296882841261676$$

$$\begin{aligned}\det M_2^3 &= -\frac{c_1^9}{1728} - \frac{c_1^7 c_2}{720} + \frac{7c_1^6 c_3}{720} - \frac{c_1^5 c_2^2}{160} + \frac{c_1^5 c_4}{120} + \frac{c_1^4 c_2 c_3}{144} - \frac{c_1^4 c_5}{24} - \frac{c_1^3 c_2^3}{144} \\ &\quad + \frac{c_1^3 c_2 c_4}{12} - \frac{c_1^3 c_6}{18} - \frac{c_1^2 c_2^2 c_3}{16} + \frac{c_1^2 c_2 c_5}{20} + \frac{c_1^2 c_3 c_4}{24} + \frac{c_1 c_2^4}{64} - \frac{c_1 c_2 c_3^2}{18} \\ &\quad + \frac{c_1 c_3 c_5}{15} - \frac{c_1 c_4^2}{16} + \frac{c_2^3 c_3}{144} - \frac{c_2^2 c_5}{40} + \frac{c_2 c_3 c_4}{24} - \frac{c_3^3}{54} + \frac{c_3 c_6}{18} - \frac{c_4 c_5}{20} \\ &=: \Phi_{25}(c_1, c_2, c_3, c_4, c_5, c_6)\end{aligned} \tag{26}$$

$$\max\left|\det M_2^3\right| \approx 0.6546327601175532$$

$$\begin{aligned}\det M_2^2 &= -\frac{11c_1^{10}}{86400} - \frac{c_1^8 c_2}{1920} + \frac{c_1^7 c_3}{360} - \frac{7c_1^6 c_2^2}{2880} + \frac{7c_1^6 c_4}{1440} - \frac{7c_1^5 c_5}{600} - \frac{c_1^4 c_2^3}{576} + \frac{c_1^4 c_2 c_4}{32} \\ &\quad - \frac{c_1^4 c_3^2}{144} - \frac{5c_1^4 c_6}{144} - \frac{c_1^3 c_2^2 c_3}{72} + \frac{c_1^3 c_2 c_5}{20} + \frac{c_1^3 c_3 c_4}{36} + \frac{c_1^2 c_2^4}{384} \\ &\quad - \frac{c_1^2 c_2^2 c_4}{32} - \frac{c_1^2 c_2 c_3^2}{24} - \frac{c_1^2 c_2 c_6}{24} + \frac{c_1^2 c_3 c_5}{15} - \frac{c_1^2 c_4^2}{32} + \frac{c_1 c_2^3 c_3}{36} \\ &\quad + \frac{c_1 c_2^2 c_5}{40} - \frac{c_1 c_3^3}{54} + \frac{c_1 c_3 c_6}{18} - \frac{c_1 c_4 c_5}{20} - \frac{c_2^5}{384} - \frac{c_2^3 c_4}{96} + \frac{c_2^2 c_3^2}{144} \\ &\quad - \frac{c_2^2 c_6}{48} + \frac{c_2 c_4^2}{32} - \frac{c_3^2 c_4}{72} + \frac{c_4 c_6}{24} - \frac{c_5^2}{25} \\ &=: \Phi_{26}(c_1, c_2, c_3, c_4, c_5, c_6)\end{aligned} \tag{27}$$

$$\max\left|\det M_2^2\right| \approx 0.5881062366350213$$

$$\begin{aligned}\det M_2^1 &= -\frac{c_1^{11}}{86400} - \frac{c_1^9 c_2}{17280} + \frac{c_1^8 c_3}{2880} - \frac{c_1^7 c_2^2}{2880} + \frac{c_1^7 c_4}{1440} - \frac{7c_1^6 c_5}{3600} - \frac{c_1^5 c_2^3}{2880} \\ &\quad + \frac{c_1^5 c_2 c_4}{160} - \frac{c_1^5 c_3^2}{720} - \frac{c_1^5 c_6}{144} - \frac{c_1^4 c_2^2 c_3}{288} + \frac{c_1^4 c_2 c_5}{80} + \frac{c_1^4 c_3 c_4}{144} + \frac{c_1^3 c_2^4}{1152} \\ &\quad - \frac{c_1^3 c_2^2 c_4}{96} - \frac{c_1^3 c_2 c_3^2}{72} - \frac{c_1^3 c_2 c_6}{72} + \frac{c_1^3 c_3 c_5}{45} - \frac{c_1^3 c_4^2}{96} + \frac{c_1^2 c_2^3 c_3}{72} + \frac{c_1^2 c_2^2 c_5}{80} \\ &\quad - \frac{c_1^2 c_3^3}{108} + \frac{c_1^2 c_3 c_6}{36} - \frac{c_1^2 c_4 c_5}{40} - \frac{c_1 c_2^5}{384} - \frac{c_1 c_2^3 c_4}{96} + \frac{c_1 c_2^2 c_3^2}{144} - \frac{c_1 c_2^2 c_6}{48} \\ &\quad + \frac{c_1 c_2 c_4^2}{32} - \frac{c_1 c_3^2 c_4}{72} + \frac{c_1 c_4 c_6}{24} - \frac{c_1 c_5^2}{25} + \frac{c_2^4 c_3}{576} + \frac{c_2^3 c_5}{80} \\ &\quad - \frac{c_2^2 c_3 c_4}{48} + \frac{c_2 c_3^3}{108} - \frac{c_2 c_3 c_6}{36} + \frac{c_2 c_4 c_5}{40} + \frac{c_3^2 c_5}{45} - \frac{c_3 c_4^2}{48} \\ &=: \Phi_{27}(c_1, c_2, c_3, c_4, c_5, c_6)\end{aligned} \tag{28}$$

$$\max\left|\det M_2^1\right| \approx 0.2952999828856822$$

$$
\begin{aligned}
\det M_3^3 = {} & -\frac{7c_1^8}{2880} - \frac{c_1^6c_2}{360} + \frac{c_1^5c_3}{36} - \frac{c_1^4c_2^2}{96} + \frac{c_1^3c_2c_3}{18} \\
& - \frac{c_1^3c_5}{10} - \frac{c_1^2c_2^3}{24} + \frac{c_1^2c_2c_4}{8} + \frac{c_1^2c_3^2}{36} - \frac{c_1^2c_6}{12} \\
& - \frac{c_1c_2^2c_3}{12} + \frac{c_1c_2c_5}{10} - \frac{c_2^4}{192} - \frac{c_2c_3^2}{36} + \frac{c_2c_6}{12} - \frac{c_4^2}{16} \\
=: {} & \varPhi_{28}(c_1, c_2, c_3, c_4, c_5, c_6)
\end{aligned} \tag{29}
$$

$$\max\left|\det M_3^3\right| \approx 0.744687558245012$$

$$
\begin{aligned}
\det M_3^1 = {} & -\frac{c_1^{10}}{17280} - \frac{c_1^8c_2}{5760} + \frac{c_1^7c_3}{720} - \frac{c_1^6c_2^2}{960} + \frac{c_1^6c_4}{720} + \frac{c_1^5c_2c_3}{720} - \frac{c_1^5c_5}{120} - \frac{c_1^4c_2^3}{576} \\
& + \frac{c_1^4c_2c_4}{48} - \frac{c_1^4c_6}{72} - \frac{c_1^3c_2^2c_3}{48} + \frac{c_1^3c_2c_5}{60} + \frac{c_1^3c_3c_4}{72} + \frac{c_1^2c_2^4}{128} \\
& - \frac{c_1^2c_2c_3^2}{36} + \frac{c_1^2c_3c_5}{30} - \frac{c_1^2c_4^2}{32} + \frac{c_1c_2^3c_3}{144} - \frac{c_1c_2^2c_5}{40} + \frac{c_1c_2c_3c_4}{24} \\
& - \frac{c_1c_3^3}{54} + \frac{c_1c_3c_6}{18} - \frac{c_1c_4c_5}{20} + \frac{c_2^5}{384} - \frac{c_2^2c_6}{24} + \frac{c_2c_3c_5}{30} \\
& + \frac{c_2c_4^2}{32} - \frac{c_3^2c_4}{36} \\
=: {} & \varPhi_{29}(c_1, c_2, c_3, c_4, c_5, c_6)
\end{aligned} \tag{30}
$$

$$\max\left|\det M_3^1\right| \approx 0.31740723754842565$$

$$
\begin{aligned}
\det M_1^1 = {} & -\frac{c_1^{12}}{1036800} - \frac{c_1^{10}c_2}{172800} + \frac{c_1^9c_3}{25920} - \frac{c_1^8c_2^2}{23040} + \frac{c_1^8c_4}{11520} - \frac{c_1^7c_5}{3600} - \frac{c_1^6c_2^3}{17280} \\
& + \frac{c_1^6c_2c_4}{960} - \frac{c_1^6c_3^2}{4320} - \frac{c_1^6c_6}{864} - \frac{c_1^5c_2^2c_3}{1440} + \frac{c_1^5c_2c_5}{400} + \frac{c_1^5c_3c_4}{720} + \frac{c_1^4c_2^4}{4608} \\
& - \frac{c_1^4c_2^2c_4}{384} - \frac{c_1^4c_2c_3^2}{288} - \frac{c_1^4c_2c_6}{288} + \frac{c_1^4c_3c_5}{180} - \frac{c_1^4c_4^2}{384} + \frac{c_1^3c_2^3c_3}{216} + \frac{c_1^3c_2^2c_5}{240} \\
& - \frac{c_1^3c_3^3}{324} + \frac{c_1^3c_3c_6}{108} - \frac{c_1^3c_4c_5}{120} - \frac{c_1^2c_2^5}{768} - \frac{c_1^2c_2^3c_4}{192} + \frac{c_1^2c_2^2c_3^2}{288} - \frac{c_1^2c_2^2c_6}{96} \\
& + \frac{c_1^2c_2c_4^2}{64} - \frac{c_1^2c_3^2c_4}{144} + \frac{c_1^2c_4c_6}{48} - \frac{c_1^2c_5^2}{50} + \frac{c_1c_2^4c_3}{576} + \frac{c_1c_2^3c_5}{80} \\
& - \frac{c_1c_2^2c_3c_4}{48} + \frac{c_1c_2c_3^3}{108} - \frac{c_1c_2c_3c_6}{36} + \frac{c_1c_2c_4c_5}{40} + \frac{c_1c_3^2c_5}{45} - \frac{c_1c_3c_4^2}{48} \\
& - \frac{c_2^6}{1536} - \frac{c_2^4c_4}{768} + \frac{c_2^3c_3^2}{864} + \frac{c_2^3c_6}{96} - \frac{c_2^2c_3c_5}{60} - \frac{c_2^2c_4^2}{128} + \frac{c_2c_3^2c_4}{48} \\
& + \frac{c_2c_4c_6}{48} - \frac{c_2c_5^2}{50} - \frac{c_3^4}{162} - \frac{c_3^2c_6}{54} + \frac{c_3c_4c_5}{30} - \frac{c_4^3}{64} \\
=: {} & \varPhi_{30}(c_1, c_2, c_3, c_4, c_5, c_6)
\end{aligned} \tag{31}
$$

$$\max\left|\det M_1^1\right| \approx 0.24689140987222155$$

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$$
\begin{aligned}
\det M &= \det H_{4,1}(f)\\
&= \frac{c_1^{12}}{1036800} - \frac{c_1^{10}c_2}{172800} - \frac{c_1^9c_3}{25920} + \frac{c_1^8c_2^2}{23040} + \frac{c_1^8c_4}{11520} + \frac{c_1^7c_5}{3600} - \frac{c_1^6c_2^3}{17280}\\
&\quad - \frac{c_1^6c_2c_4}{960} + \frac{c_1^6c_3^2}{4320} - \frac{c_1^6c_6}{864} + \frac{c_1^5c_2^2c_3}{1440} + \frac{c_1^5c_2c_5}{400} + \frac{c_1^5c_3c_4}{720} - \frac{c_1^4c_2^4}{4608}\\
&\quad - \frac{c_1^4c_2^2c_4}{384} - \frac{c_1^4c_2c_3^2}{288} + \frac{c_1^4c_2c_6}{288} - \frac{c_1^4c_3c_5}{180} + \frac{c_1^4c_4^2}{384} + \frac{c_1^3c_2^3c_3}{216} - \frac{c_1^3c_2^2c_5}{240}\\
&\quad + \frac{c_1^3c_3^3}{324} + \frac{c_1^3c_3c_6}{108} - \frac{c_1^3c_4c_5}{120} - \frac{c_1^2c_2^5}{768} + \frac{c_1^2c_2^3c_4}{192} - \frac{c_1^2c_2^2c_3^2}{288}\\
&\quad - \frac{c_1^2c_2^2c_6}{96} + \frac{c_1^2c_2c_4^2}{64} - \frac{c_1^2c_3^2c_4}{144} - \frac{c_1^2c_4c_6}{48} + \frac{c_1^2c_5^2}{50} - \frac{c_1c_2^4c_3}{576}\\
&\quad + \frac{c_1c_2^3c_5}{80} - \frac{c_1c_2^2c_3c_4}{48} + \frac{c_1c_2c_3^3}{108} + \frac{c_1c_2c_3c_6}{36} - \frac{c_1c_2c_4c_5}{40} - \frac{c_1c_3^2c_5}{45}\\
&\quad + \frac{c_1c_3c_4^2}{48} + \frac{c_2^6}{1536} - \frac{c_2^4c_4}{768} + \frac{c_2^3c_3^2}{864} - \frac{c_2^3c_6}{96} + \frac{c_2^2c_3c_5}{60} + \frac{c_2^2c_4^2}{128}\\
&\quad - \frac{c_2c_3^2c_4}{48} + \frac{c_2c_4c_6}{48} - \frac{c_2c_5^2}{50} + \frac{c_3^4}{162} - \frac{c_3^2c_6}{54} + \frac{c_3c_4c_5}{30} - \frac{c_4^3}{64}\\
&=: \Phi_{31}(c_1,c_2,c_3,c_4,c_5,c_6)
\end{aligned} \tag{32}
$$

$$\max|\det M| = \max|\det H_{4,1}(f)| \approx 0.12499999997131443$$

5. Conclusions. Based on the applied method of differential evolution, we can formulate the following theorem:

THEOREM 5.1. *If $f \in \mathcal{S}^*$, then*

$$|\det H_{4,1}(f)| \le \frac{1}{8} = 0.125. \tag{5.1}$$

The inequality (5.1) is sharp and equality holds for the function $f \in \mathcal{S}^*$ defined by

$$\frac{zf'(z)}{f(z)} = \frac{1+z^4}{1-z^4}, \quad z \in \mathbb{D},$$

and its rotations, i.e., for

$$f(z) = \frac{z}{\sqrt{1-z^4}}, \quad z \in \mathbb{D},\ \sqrt{1} := 1,$$

and its rotations.

The following results are known for the class $\mathcal{S}^*$ of starlike functions:

THEOREM 5.2. *If $f \in \mathcal{S}^*$ is of the form (1.1), then*

$$|a_n| \le n, \qquad n \in \mathbb{Z}_2; \tag{5.2}$$

$$|J_{m,n}(f)| = |a_{m+n-1} - a_n a_m| \le (m-1)(n-1), \quad m,n \in \mathbb{Z}_2; \tag{5.3}$$

$$|J_{3,3,2}(f)| = |a_2a_4 - a_3^2| \le 1; \tag{5.4}$$

$$|\det H_{3,1}(f)| \le \frac{4}{9}. \tag{5.5}$$

All inequalities are sharp.

REMARK 5.3. The sharp estimates (5.2) were found by Nevanlinna [33] (cf. [13, p. 44]). The estimate (5.3) was demonstrated by Ma [32], who thus confirmed the generalized Zalcman conjecture for starlike functions. The sharp estimate in (5.4) is found in [3]. The

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sharp inequality (5.5) was shown by Kowalczyk et al. [22], solving a long-standing problem. In the paper [2], Babalola showed that $|\det H_{3,1}(f)| \leq 16$. Next, Zaprawa [44] improved Babalola's result by deriving $|\det H_{3,1}(f)| \leq 1$. This result was improved by Kwon et al. [25], where it was verified that $|\det H_{3,1}(f)| \leq 8/9$. In [26], Kwon and Sim proved that $-4/9 \leq \det H_{3,1}(f) \leq \sqrt{3}/9$ for starlike functions having real coefficients and that both inequalities are sharp. Further, Zaprawa et al. [45] verified that for the whole class $\mathcal{S}^*$, $|\det H_{3,1}(f)| \leq 5/9$.

REMARK 5.4. Let us now emphasize that the results obtained by the evolutionary algorithm are consistent (in the sense of numerical approximation) with those well-known results recalled in Theorem 5.2. Results in points (2)–(7) are consistent with (5.2) for $n = 2, \ldots, 7$. The result of (8) is consistent with (5.3). The results of (9) and (22) are consistent with (5.4) and (5.5), respectively.

Let us note that the results for the functional $J_{m,n,k}$, obtained successively in points (14)–(21) by applying the evolutionary algorithm, suggest to formulate the following general conjecture for starlike functions. It seems that this conjecture may also hold true for the whole class $\mathcal{S}$.

CONJECTURE 5.5. *Let $k \in \mathbb{N}$, $m, n \in \mathbb{Z}_{k+1}$. If $f \in \mathcal{S}$ $(\mathcal{S}^*)$, then*

$$|J_{m,n,k}(f)| \leq (m-k)(n-k),$$

with equality for the Koebe function and its rotations.

In addition, the study of the functional $J_{m,n,k}$ on known subclasses in the class $\mathcal{A}$ seems to be a sensible and interesting problem.

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