

A Simple Hausdorff Space which is Lindelöf, First Countable, CCC but not Separable

By

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Abstract

This short note presents a simple example of a Hausdorff space which has the Lindelöf property, satisfies the countable chain condition and where every point has a countable neighbourhood base but which is not separable.

Key words: Lindelöf space, countable neighbourhood base, countable chain condition, separable, MSC 1991: 54G15.

1. Introduction

Countability properties play an essential role in set theoretic topology and have been studied from its beginning (cf. for instance [2] and [4]). This short note presents a very simple example of a topological space X with the properties stated in the title. A more advanced example of this type has been given by M. G. Bell in [1]. Bell uses several nontrivial constructions and results from set theoretic topology to show that his space has the desired properties and even several more remarkable ones. Nevertheless it seems to be of some interest also to present our example: Beside very elementary topology it does not need more than the knowledge of the first uncountable ordinal and its most fundamental properties. Construction and proof therefore do not exceed the level of most

undergraduate courses in general topology. The note might be considered as a small additional remark to the famous book of L.A. Steen and J.A. Seebach [3] where no space of this type can be found.

2. Construction of the Space

Let $\Omega = \{\alpha \mid \alpha < \Omega\}$ denote the first uncountable ordinal number. For the rest we fix a family $A_\alpha, \alpha \in \Omega$, of pairwise disjoint, countable dense subsets of the (open) unit interval $(0,1)$.

Such a family exists: As an example consider the factor group \mathbf{R}/\mathbf{Q} of the additive group \mathbf{R} of the reals by the subgroup \mathbf{Q} of the rationals. \mathbf{R}/\mathbf{Q} has the cardinality of the continuum, hence $|\mathbf{R}/\mathbf{Q}| \geq \Omega$. For every subfamily $B_\alpha, \alpha \in \Omega$, (B_α pairwise distinct) of \mathbf{R}/\mathbf{Q} the sets $\mathbf{A}_\alpha = \mathbf{B}_\alpha \cap (0, 1), \alpha < \Omega$, do the job.

Now consider the set

$$X = \{(x, \alpha) \mid x \in A_\alpha, \alpha < \Omega\}.$$

It is obvious that the system of all

$$O(a, b, \alpha) = \{(x, \beta) \in X \mid a < x < b, \alpha \leq \beta < \Omega\}$$

with $0 \leq a < b \leq 1$ and $\alpha < \Omega$ is the base of a topology on X .

3. Proof of the Properties

X is a **Hausdorff** space: For two distinct points $(x_i, \alpha_i) \in X, i = 1, 2$, we have by construction $x_1 \neq x_2$, w.l.o.g. $x_1 < x_2$. Then $O(0, x, \alpha_1)$ and $O(x, 1, \alpha_2)$ with any $x \in (x_1, x_2)$ are disjoint neighbourhoods.

X has the **Lindelöf** property, i.e. every open covering has a countable subcovering: Let $O_i, i \in I$, be an arbitrary open covering of X , i.e. all O_i are open and their union is X . We have to construct a countable subcovering. W.l.o.g. we may assume that all O_i are base sets: $O_i = O(a_i, b_i, \alpha_i)$. Let $\mathcal{A} \subseteq (0,1)$ denote the union of all $A_\alpha, \alpha < \Omega$. Since $(0,1)$ has a countable topological base the same holds for its subspace \mathcal{A} which therefore has the Lindelöf property. Thus the open covering $U_i = (a_i, b_i) \cap \mathcal{A}, i \in I$, of \mathcal{A} contains a countable subcovering. $U_{i_n}, n \in \mathbf{N}$, of \mathcal{A} . Let $\alpha_0 < \Omega$ denote the least upper bound of the corresponding α_{i_n} . Since the supremum is taken over a countable set of at most countable ordinal numbers, α_0 is at most countable itself. An immediate consequence is that the difference set

$$D = X \setminus \bigcup_{n \in \mathbf{N}} O_{i_n} \subseteq \bigcup_{\alpha < \alpha_0} A_\alpha \times \{\alpha\}$$

is at most countable, say $D \subseteq \{d_1, d_2, \dots\}$. Take $O_{i,n}$ in such a way that $d_n \in O_{i,n}$ to get a countable subcovering consisting of all O_m and $O_{i,n}$, $n \in \mathbf{N}$.

X is **first countable**, i.e. every point has a countable neighbourhood base: For every $(x, \alpha) \in X$ a countable neighbourhood base is given by the sets $O(y_n, z_n, \alpha)$, $n \in \mathbf{N}$, if the intervals (y_n, z_n) form a neighbourhood base of x in $(0,1)$.

X satisfies the countable chain condition **CCC**, i.e. every family of pairwise disjoint nonempty open sets in X is at most countable: Let O_i , $i \in I$, be any family of pairwise disjoint nonempty open sets in X . We want to derive a contradiction from the assumption that I is not countable. Again we may suppose $O_i = O(a_i, b_i, \alpha_i)$. Since the unit interval $(0,1)$ has a countable topological base, it satisfies the countable chain condition. Hence the family (a_i, b_i) , $i \in I$, cannot be pairwise disjoint, say $(a, b) \subseteq (a_{i_1}, b_{i_1}) \cap (a_{i_2}, b_{i_2})$. Let $\alpha = \max_{j=1,2} \alpha_{i_j}$. A_x is dense in $(0,1)$, thus we can find a $y \in A_x \cap (a, b)$ yielding $(y, \alpha) \in O_{i_1} \cap O_{i_2}$, contradiction.

X is **not separable**: If $M = \{(x_n, \alpha_n) \mid n \in \mathbf{N}\} \subseteq X$ is an arbitrary countable subset of X and $\alpha = \sup_{n \in \mathbf{N}} \alpha_n + 1$, then it is obvious that the set $O(0, 1, \alpha)$ is a nonempty open set in X which does not intersect M .

References

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