On Optimal Liftings of Sequences*

Von

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Edmund Hlawka zum achtzigsten Geburtstag gewidmet

(Vorgelegt in der Sitzung der math.-nat. Klasse am 12. Dezember 1996
durch das w. M. August Florian)

1. Setting of the Problem

From the early work of Hlawka and Korobow on it became clear that many problems of high dimensional numerics are to be solved using number theoretical methods. NTN (Number Theoretical Numerics) provides best possible methods for simulation, numerical integration, approximation and interpolation, integral equations and many other problems in the multivariate domain, where uniform distribution of sequences or multivariate integration plays the key role (approximation of functions e.g. is mostly based on convolution). For a review see for example [1], [4], [5], [2], [9]. Recently NTN plays a role in image processing too [21] and [23]. It is considered as an important problem to construct sequences in the s-dimensional unit cube $G_s = [0,1)^s$ having good properties from the simulation or numerical integration point of view. It means essentially the quality of uniform distribution of the sequences to be used. The quality of distribution of sequences can be measured by means of different concepts, of which the most commonly used are the Discrepancy D_N^* and the Diaphony F_N of the sequence $\underline{x}_0$,

$\underline{x}_1, \ldots, \underline{x}_{N-1} \in G_s = [0,1]^s$. D_N^* and F_N are knowingly defined by

$$D_N^*(w_N) = \sup_{\substack{0 \le \gamma_i < 1 \\ i=1,\ldots,s}} \left| \frac{1}{N} \sum_{k=0}^{N-1} \chi_{\underline{\gamma}}(\underline{x}_{k-1}) - \gamma_1 \ldots \gamma_s \right| \tag{1}$$

$$F_N^2(w_N) = \frac{1}{N^2} \sum_{k,l=1}^{N} H_2(\underline{x}_k - \underline{x}_l) - 1 \tag{2}$$

where

$$H_2(\underline{x}) = \sum_{\underline{m} \in Z^s} \frac{1}{(\bar{m}_1 \ldots \bar{m}_s)^2} e^{2\pi i \underline{m}\underline{x}} \tag{3}$$

is the s-variate normed Bernoulli-Polynomial modulo 1,

$$H_2(\underline{x}) = \prod_{i=1}^{s} \left(1 - \frac{\pi^2}{6} + \frac{\pi^2}{2}(1 - 2x_i)^2 \right), \tag{4}$$

$\bar{m} = \max(1, |m|), m \in \mathbf{Z}, \underline{x} = (x_1, \ldots, x_s), x_i \in \mathbf{R}$. For properties and use of the Diaphony see [15], [9], [25]. The typical order of magnitude of D_N^* and F_N of good sequences $\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{N-1} \in G_s$ is

$$D_N^* \ll \frac{\ln^{\beta_1} N}{N}, \quad F_N = \frac{\ln^{\beta_2} N}{N}, \quad \beta_1, \beta_2 > 0. \tag{5}$$

As usual $a \ll b$ means the existence of a positive number c such that $|a| \le cb$.

Recently the Salzburg research groups of Hellekalek and Larcher and the Vienna group of Prof. Niederreiter received best possible results with respect to the estimation of D_N^*, F_N and the integration error and for Quasi-Monte Carlo methods.

In many cases of application of high dimensional integration or simulation a more complicated problem is posed. It consists in the simplest case of the following problem:

Given numbers $x_0, \ldots, x_{N-1} \in G_1 = [0,1)$, one has to construct a set of points $\underline{x}_k = (x_{1k}, \ldots, x_{sk}) \in G_s = [0,1)^s$, $k = 0, \ldots, N-1$ with good distribution properties, such that all the $x_{ik} \in \{x_0, \ldots, x_{N-1}\}$, $i = 1, \ldots, s$, $k = 0, \ldots, N-1$. It follows from the definition of Discrepancy D_N^*, that the Discrepancy of such a sequence $\underline{x}_k, k = 0, \ldots, N-1$, in general cannot be less than the Discrepancy of the original $x_0, \ldots, x_{N-1} \in G_1$. So the problem consists of optimal Lifting of the sequence $x_0, \ldots, x_{N-1} \in G_1$ to $G_s = [0,1)^s$. In the theory of optimal experiments the problem is well known too. So this paper is also a contribution to the theory of uniform designs.

2. Solutions of the Lifting Problem

Let $\Gamma_N = \{0, 1, \ldots, N-1\}$ and let for $j = 1, \ldots, s$ be given the finite sequences $w_j = w_j(N) = (x_{0,j}, x_{2,j}, \ldots, x_{N-1,j}) \in [0,1)^N$. The sequences w_j have discrepancies $D_N^*(w_j) = D_{N,j}^*, j = 1, \ldots, s$. Let $\Omega = \Pi_{j=1}^s w_j$ such that Ω consists of all points $\underline{x} = (x(k_1, 1), x(k_2, 2), \ldots, x(k_s, s))$, where $x(k_j, j), = x_{k_j, j}, \quad (k_1, \ldots, k_s) \in \Gamma_N^s$. Let $G_s = [0,1)^s$ and $p_j(x_1, \ldots, x_s) = x_j, j = 1, \ldots, s$, be the projection of G_s onto $[0,1)$. We give the following

Definition 1. *The function $L : (w_1, \ldots, w_s) \to \Omega$ such that*

a) $L(w_1, \ldots, w_s) = \{\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N\} = \underline{w}(N)$ *is a sequence of N points in G_s and such that*

b) $p_j(L(w_1, \ldots, w_s)) = \{x_{1j}, \ldots, x_{Nj}\}$ *for $j = 1, \ldots, s$.*

is called a Lifting of the one-dimensional sequences $w_1, \ldots, w_s$ to G_s. The Lifting L will be called regular if

b') $p_j(L(w_1, \ldots, w_s)) = P_j(w_j)$, *where P_j is a permutation of w_j (and of G_N) depending on j and L itself.*

We will prove now the following

Theorem 1. *Let $\underline{a} = (a_1, \ldots, s_s) \in \mathbf{Z}^s$ and $(a_j, N) = 1$ for $j = 1, \ldots, s$. Let furthermore $D_N^*(\underline{a})$ be the Discrepancy of the sequence $k\underline{a}/N$, $k = 0, \ldots, N-1$ and consider the regular lifting $L_{\underline{a}}$ induced by $\underline{a}$*

$$L_a(w_1, \ldots, w_s) = \{\underline{x}(\underline{a}k), \ k = 0, \ldots, N-1\} = w(\underline{a}) \qquad (6)$$

such that $\underline{x}(\underline{a}k) = (x(a_1k, 1), x(a_2k, 2), \ldots, x(a_sk, s))$, $k = 0, \ldots, N-1$, (we take the indexes $a_jk \mod N$ of course), than the Discrepancy $D_N^*(w(\underline{a}))$ can be estimated by

$$\max_{j=1, \ldots, s} D_N^*(w_j) \leq D_N^*(w(\underline{a})) \ll \sum_{j=1}^s D_N^*(w_j) + D_N^*(\underline{a}) \qquad (7)$$

Proof: Without loss of generality we assume the $w_j, j = 1, \ldots, s$, being ordered: $0 \leq x_{0,j} \leq x_{1,j} \leq \ldots \leq x_{N-1,j} < 1$. Let $\gamma = (\gamma_1, \ldots, \gamma_s)$, $0 < \gamma_j \leq 1, j = 1, \ldots, s$ and let $I(\gamma) = \{\underline{x} : 0 \leq x_j < \gamma_j, j = 1, \ldots, s\} \subseteq G_s$. Let for $\underline{x} \in G_s$

$$\chi(\underline{x}) = \begin{cases} 1 & \underline{x} \in I(\gamma) \\ 0 & \underline{x} \notin I(\gamma) \end{cases} \qquad (8)$$

and let for $\underline{x} = (x(k_1, 1), \ldots, x(k_s, s)) \in \Omega$ the function ψ be defined by

$$\psi(k_1, \ldots, k_s) = \begin{cases} 1 & \text{if} \quad \underline{x} \in I(\gamma) \\ 0 & \text{if} \quad \underline{x} \notin I(\gamma) \end{cases} \qquad (9)$$

such that $\psi(k_1,\ldots,k_s) = \chi(x(k_1,1),\ldots,x(k_s,s))$. According to the assumed ordering of the points of w_j, there are unique integers $n_j, 0 \leq n_j \leq N-1$, such that $x_{n_j,j} < \gamma_j$ and $x_{n_j+1,j} \geq \gamma_j, j=1,\ldots,s$. So we get for the Fourier-transform $\hat{\psi}$ of $\psi:\Gamma_N^s \to \{0,1\}$

$$\hat{\psi}(m_1,\ldots,m_s) = \frac{1}{N^s}\sum_{k_1=0}^{n_1}\cdots\sum_{k_s=0}^{n_s} e^{-2\pi i(m_1 k_1 + \ldots + m_s k_s)/N}, \qquad (10)$$

where the dual group of G_N^s is represented by

$$\widehat{\Gamma_N^s} = \{(m_1,\ldots,m_s), -M_1 \leq m_j \leq M_2, j=1,\ldots,s\} \qquad (11)$$

with $M_1 = (N-1)/2$ for $N \equiv 1(2), M_1 = N/2 - 1$ for $N \equiv 0(2), M_2 = M_1$ for $N \equiv 1(2), \; M_2 = M_1 + 1, \; N \equiv 0(2)$. For $m_j = 0$ we have with $\bar{m} = \max(1,|m|)$

$$\left|\sum_{k_j=0}^{n_j} e^{-2\pi i m_j k_j/N}\right| = n_j + 1 \leq \frac{N}{\bar{m}_j}. \qquad (12)$$

For $m_j \neq 0$ is because of (11) $m_j \not\equiv 0 \; (N)$ and henceforth, with the notation $\ll x \gg = \min\{|x-g|, g \in \mathbf{Z}\}$,

$$\left|\sum_{k_j=0}^{n_j} e^{-2\pi i m_j k_j/N}\right| = \left|\frac{e^{2\pi i m_j(n_j+1)/N} - 1}{e^{2\pi i m_j/N} - 1}\right|$$

$$\leq \frac{1}{\sin(\pi m_j/N)} \leq \frac{1}{2 \ll \pi m_j/N \gg} \leq \frac{1}{2}\frac{N}{\bar{m}_j}. \qquad (13)$$

So, for $(m_1,\ldots,m_s) \in \widehat{\Gamma_N^s}$, holds the estimation

$$|\hat{\psi}(m_1,\ldots,m_s)| = \frac{1}{N^s}\left|\prod_{j=1}^{s}\sum_{k_j=0}^{n_j} e^{-2\pi i m_j k_j/N}\right| \leq \frac{1}{N^s}\prod_{j=1}^{s}\frac{N}{\bar{m}_j} = \frac{1}{\bar{m}_1 \ldots \bar{m}_s}. \qquad (14)$$

The technicality (14) is frequently used and well known. A direct inspection of (10) shows

$$\hat{\psi}(0,\ldots,0) = \prod_{j=1}^{s}\frac{n_j+1}{N}. \qquad (15)$$

For the L_a-lifted sequence $w(\underline{a})$, (6), holds with respect to (8), (9)

$$A := A(w(\underline{a}) \in I(\underline{\gamma})) := \sum_{k=0}^{N-1}\chi_\gamma(\underline{x}(\underline{a}k)) = \sum_{k=0}^{N-1}\psi(a_1 k,\ldots,a_s k). \qquad (16)$$

The counting function A jumps in our case of the sequence $\underline{x}(\underline{a}k) = \underline{a}k/N$, $k = 0, \ldots, N-1$, $\underline{a} \in \mathbf{Z}^s$, only at the points $\underline{\gamma} = (n_1 + 1, n_2 + 1, \ldots, n_s + 1)/N$ such that one gets immediately

$$D_N^*(\underline{a}) = \max_{\substack{0 \le n_j \le N-1 \\ j = 1, \ldots, s}} \left| \frac{A}{N} - \frac{(n_1 + 1)\ldots(n_s + 1)}{N^s} \right|. \tag{17}$$

Using e.g. the s-variate Taylor-formula of order one we get in a straight forward manner

$$\left| \prod_{j=1}^s \gamma_j - \prod_{j=1}^s \frac{n_j + 1}{N} \right| \le \sum_{j=1}^s (\prod_{\substack{k \ne j \\ k=1}} (1 + D_{N,k}^*)) D_{N,j}^*. \tag{18}$$

Combining (17) and (18) we get the right hand side part of (7)

$$D_N^*(w(\underline{a})) \le D_N^* + \sum_{j=1}^s (\prod_{\substack{k \ne j \\ k=1}} (1 + D_{N,k}^*)) D_{N,j}^*. \tag{19}$$

The left hand side part of (7) follows readily: Let $\varepsilon > 0$ and

$$D_N^*(w_{j0}) = \max_{j=1,\ldots s} D_N^*(w_j). \tag{20}$$

By the definition of Discrepancy there is an interval $[0, \gamma_{j0}(\varepsilon)) = I_{j0}(\varepsilon)$ such that

$$\left| \frac{A(w_{j0} \in I_{j0}(\varepsilon))}{N} - \gamma_{j0}(\varepsilon) \right| \ge D_{N,j0}^* - \varepsilon. \tag{21}$$

Let now

$$I_s = T \times T \ldots \times T \times I_{j0}(\varepsilon) \times T \times \ldots \times T, \tag{22}$$

then we have

$$D_N^*(w\underline{a}) \ge \left| \frac{A(w(\underline{a}) \in I_s)}{N} - \lambda(I_s) \right|$$
$$= \left| \frac{A(w_{j0} \in I_{j0}(\varepsilon))}{N} - \lambda_{j0}(\varepsilon) \right| \ge D_{N,j0}^* - \varepsilon. \tag{23}$$

The proof of Theorem 1 is complete.

There is a vast literature on the estimation of $D_N^*(\underline{a})$ if the $\underline{a}$ are good lattice points or optimal coefficients (e.g. [9]). One observes that instead of the cyclic Liftings $k \to \bar{a}k \bmod N$ one can use e.g. (t,m,s)-nets to lift sequences as well. For the definition and properties of (t,m,s)-nets see [9]. Some latest results can be found in [6]. This observation leads to the definition of Discrepancy of a Lifting:

Definition 2. *Let* $L: G_N \to G_N^s$, $L(k) = \underline{a}(k) \in G_N^s$, *be a Lifting. We call the Discrepancy of the sequence* $\underline{a}(k)/N$, $k = 0, \dots, N-1$, *mod* 1, *the Discrepancy of the Lifting* L:

$$D_N^*(L) = D_N^*\left(\frac{\underline{a}(k)}{N}, k = 0, \dots, N-1\right). \tag{24}$$

Analogously the Diaphony $F_N(L)$ *of the Lifting* L *is defined by*

$$F_N^2(L) = \frac{1}{N^2} \sum_{k=1}^N \sum_{l=1}^N H_2\left(\frac{\underline{a}(k) - \underline{a}(l)}{N}\right) - 1. \tag{25}$$

Clearly, $D_N^*(L)$ and $F_N(L)$ describe the mixing properties of the Lifting L. The following more general theorems holds:

Theorem 2. *If* $L: G_N \to G_N^s$ *is a regular Lifting, then*

$$\max_{j=1,\dots,s} D_N^*(w_j) \leq D_N^*(L(w_1, \dots, w_s)) \leq D_N^*(L)$$

$$+ \sum_{j=1}^s \left(\prod_{\substack{k \neq j \\ k=1}}^s (1 + D_{N,k}^*)\right) D_{N,j}^* \tag{26}$$

holds.

We will not carry out the proof which is similar to the proof of the previous theorem.

3.   Some Computational Aspects

In the case of application of the Lifting method one is interested in easy and fast computation of the quality of the lifted sequences and furthermore in the quality of the lift itself. According to Theorem 1 and (7) the problem consists of a fast computation or effective estimation of $D_N^*(w_j)$, $j = 1, \dots, s$ and of $D_N^*(\underline{a})$ respectively. The computation of the Discrepancy of the one-dimensional sequences $w_j, j = 1, \dots, s$ is readily performed by means of the formula [5]

$$D_N^*(w_j) = \frac{1}{2N} + \max_{\substack{k=0,\dots,N-1 \\ j=1,\dots,s}} \left| x_{k,j} - \frac{2k+1}{2N} \right|. \tag{27}$$

The computation or even the fast and effective estimation of $D_N^*(\underline{a})$ causes more problems. The following remarks, all based on known results or techniques respectively, could be helpful in various situations of applications of NTN to applied problems. From (14), (15), (17) it is clear that for the cyclic Lifting $L(\underline{a})$, $\underline{a} \in \mathbf{Z}^s$ holds $(\delta_N(g) = 0$ if $g \not\equiv 0$ (N),

$\delta_N(g) = 1$ otherwise)

$$D_N^*(\underline{a}) \le \sum_{\substack{0 \neq \underline{m} = -M_1}}^{M_2} \frac{\delta N(\underline{a}\underline{m})}{\overline{m}_1 \ldots \overline{m}_s}. \tag{28}$$

Since Bakhvalov and Korobov [4] the minimal value $q = q(\underline{a})$ of nontrivial $\overline{m}_1 \ldots \overline{m}_s$ with $\delta_N(\underline{a}\,\underline{m}) = 0$ plays a key role in the theory of good lattice points (cf. e.g. [9], p. 109) gave the effective estimation of $D_N^*(\underline{a})$ using (28):

$$D_N^* \underline{a} \le c_s \frac{(\ln N)^s}{q}. \tag{29}$$

An effective lower bound for $q = q(\underline{a})$ is provided by the Diaphony of $k\underline{a}/N$, $k = 0, \ldots, N-1$:

$$F_N^2(\underline{a}) = \sum_{k=0}^{N-1} H_2\left(\frac{k\underline{a}}{N}\right) = \sum_{\overline{m}_1 \ldots \overline{m}_s \ge q} \frac{\delta_N(\underline{a}\underline{m})}{(\overline{m}_1 \ldots \overline{m}_s)^2}, \tag{30}$$

which leads immediately to the effective estimation

$$D_N^*(\underline{a}) \le c_s F_N(\underline{a}) \cdot (\ln N)^s. \tag{31}$$

Because of the symmetry properties of $H_2(x)$ the computation of the right hand side of (31) is performed in $N/2$ steps. We propose now an other estimator for $D_N^*(\underline{a})$ which is based on techniques used by Korobov ([4]) to prove optimality conditions for integer vectors $\underline{a}$.

Theorem 3. *Let $N \in \mathbf{N}$, $\underline{a} = (a_1, \ldots, a_s) \in \mathbf{Z}^s$, $(a_i, N) = 1$ for $i = 1, \ldots, s$ then*

$$D_N^*(\underline{a}) = \frac{1}{N} \sum_{k=1}^{N-1} \left[1 - 2\ln\left(2\sin\pi\frac{a_1 k}{N}\right)\right] \ldots \left[1 - 2\ln\left(2\sin\pi\frac{a_s k}{N}\right)\right]$$

$$- 1 + R_N(\underline{a}), \tag{32}$$

with

$$|R_N(\underline{a})| \le (s+1)\frac{(3 + 2\ln N)^s}{N}. \tag{33}$$

Proof: The following Lemma is well known and easy to prove.

Lemma 1.

$$a) \quad 1 - 2\ln(2\sin\pi x) = \sum_{m=-(N-1)}^{N-1} \frac{e^{2\pi i m x}}{\overline{m}} + \frac{\theta}{N \ll x \gg}, |\theta| \le 1 \tag{34}$$

b) For $u_j = v_j + r_j, j = 1, \ldots, s$ holds

$$\prod_{j=1}^{s} u_j = \prod_{j=1}^{s} (v_j + r_j) = \prod_{j=1}^{s} v_j + \sum_{j=1}^{s} \left(\prod_{\substack{k=1 \\ k \neq j}}^{s} (v_k + \vartheta r_k) \right) r_j \qquad (35)$$

with some $|\vartheta| < 1$.

Let now

$$u_j = 1 - 2 \ln \left(2 \sin \pi \frac{a_i k}{N} \right), \qquad (36)$$

$$r_j = \sum_{m_j = -(N-1)}^{N-1} \frac{e^{2\pi i m_j a_j k/N}}{\overline{m}_j},$$

$$r_j = \frac{\theta_j}{N \ll a_j k/N \gg}, |\theta| \leq 1, j = 1, \ldots, s.$$

We have

$$|u_j| \leq 1 + 2 |\ln \left(2 \sin \frac{\pi}{N} \right)| \leq 1 + 2 \ln N, \qquad (37)$$

$$|r_j| \leq \frac{1}{N \ll 1/N \gg} = 1.$$

Consequently holds for $k = 1, \ldots, N-1$

$$\left[1 - 2 \ln \left(2 \sin \pi \frac{a_1 k}{N} \right) \right] \ldots \left[1 - 2 \ln \left(2 \sin \pi \frac{a_s k}{N} \right) t \right] \qquad (38)$$

$$= \sum_{m_1, \ldots, m_s = -(N-1)}^{N-1} \frac{e^{2\pi i (a_1 m_1 + \ldots + a_s m_s) k/N}}{\overline{m}_1 \ldots \overline{m}_s} + \theta_0 \frac{(2 + 2 \ln N)^{s-1}}{N}.$$

$$\cdot \left(\frac{|\theta_1|}{\ll a_1 k/N \gg} + \ldots \frac{|\theta_s|}{\ll a_1 k/N \gg} \right).$$

Obviously holds

$$\sum_{k=1}^{N-1} e^{2\pi i m k/N} = -1 + \sum_{k=1}^{N-1} e^{2\pi i m k/N} = N \delta_N(m) - 1 \qquad (39)$$

such that we get

$$\sum_{k=1}^{N-1} [1 - 2\ln(2\sin\pi a_1 k/N)]\dots[1 - 2\ln(2\sin\pi a_s k/N)] \qquad (40)$$

$$= \sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{N\vartheta_N(a_1 m_1 + \dots a_s m_s) - 1}{\overline{m}_1\dots\overline{m}_s} + \frac{(2 + 2\ln N)^{s-1}}{N}$$

$$\cdot \sum_{k=1}^{N-1} \theta_0\left(\frac{\theta_1}{\ll a_1 k/N\gg} + \dots \frac{\theta_s}{\ll a_s k/N\gg}\right)$$

$$= N + N \sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{\delta_N(a_1 m_1 + \dots a_s m_s)}{\overline{m}_1\dots\overline{m}_s} \sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{1}{\overline{m}_1\dots\overline{m}_s}$$

$$+ \frac{(2 + 2\ln N)^{s-1}}{N} \sum_{k=1}^{N-1} \theta_0\left(\frac{a_1}{\ll a_k/N\gg}\dots\frac{a_1}{\ll a_k/N\gg}\right).$$

Using

$$\sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{1}{\overline{m}_1\dots\overline{m}_s} \le (3 + 2\ln N)^s, \qquad (41)$$

$$\sum_{k=1}^{N-1} \frac{1}{\ll a_j k/N\gg} \le 2N(1 + \ln N)$$

we get finally

$$\sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{\delta_N(a_1 m_1 + \dots + a_s m_s)}{\overline{m}_1\dots\overline{m}_s} = \frac{1}{N}\sum_{k=1}^{N-1}\left[1 - 2\ln\left(2\sin\pi\frac{a_1 k}{N}\right)\right]\dots$$
$$\qquad (42)$$

$$\left[1 - 2\ln\left(2\sin\pi\frac{a_s k}{N}\right)\right] + R_N(\underline{a}),$$

where

$$|R_N| \le \left|\frac{1}{N}\sum_{m_1,\dots,m_s = -(N-1)}^{N-1} \frac{1}{\overline{m}_1\dots\overline{m}_s} + (2 + 2\ln N)^{s-1} s\, 2N(1 + \ln)\frac{1}{N^2}\right|$$
$$\qquad (43)$$

$$\le \frac{(3 + 2\ln N)^s}{N} + s\frac{(2 + 2\ln N)^{s-1}}{N}(1 + \ln N) \le \frac{(3 + 2\ln N)^s}{N}(s + 1).$$

Because of the well known inequality (confer in this paper (28))

$$D_N^*(\underline{a}) \le \sum_{m_1,\ldots,m_s = -(N-1)}^{N-1} \frac{\delta_N(a_1 m_1 + \ldots + a_s m_s)}{\overline{m}_1 \ldots \overline{m}_s} \tag{44}$$

the proof of the theorem is now complete.

Remark. Despite the fact that for certain N, s essentially best possible lattice points $\underline{a} = (a_1, \ldots, a_s)$ and corresponding estimations of $D_N^*(a)$ are known, the above theorem is useful, because it is not always possible to use (or even to find) the best possible lattice points. Frequently one has to use some Lifting $L_{\underline{a}}$, where the $\underline{a}$ comes from any where. The complexity of the right hand side of (44) is N^s, whereas the complexity of the right hand side of (42) is only $N/2$.

Peter Zinterhof jun. wrote an efficient C-code for cyclic liftings according to [6]. It works extremely fast on shared memory machines using the software PVM (Parallel Virtual Maschine).

Zinterhof jun. made also a series of experiments. We now give some numerical results as examples. Because of the high computational complexity of Discrepancy in higher dimensions we use only the Diaphony as estimator. Confer however (31), (32). For the convenience of notation let

$$l_N = \left| \frac{1}{N} \sum_{k=1}^{N-1} \left[1 - 2\ln\left(2\sin\pi\left\{ \frac{a_1 k}{N} \right\} \right) \right] \cdot \left[1 - 2\ln\left(2\sin\pi\left\{ \frac{a_2 k}{N} \right\} \right) \right] - 1 \right|. \tag{45}$$

We call the expression $l_N(\underline{x}_0, \ldots, \underline{x}_{N-1})$ the Logophony of the finite sequence $\underline{x}_0, \ldots, \underline{x}_{N-1}$. It will be studied more extensively in a forthcoming paper.

We mention in this paper only some results in the two-dimensional case. In this case we use the fact that cyclic liftings based on number theoretic properties of the Fibonacci-sequence are a very good choice. So we call those Liftings Fibonacci-Liftings. We will use the classical Fibonacci-sequence $1, 1, 2, 3, 5, \ldots, f_n = f_{n-1} + f_{n-2}, \ldots$. For the Fibonacci-Lifting as we call it choose $N = f_n, a_1 = 1, a_2 = f_{n-1}$. Uniform distribution properties of the triplets $(N, 1, a_2)$ are apparently described the first time in [4]. See also Niederreiter [9]. We will give now two simple numerical examples

a) Lifting the sequence $k/N, k = 1, \ldots, N$ by the two-dimensional Fibonacci-Lifting. Let F_1 be the Diaphony of the sequence k/N itself and let F_2 be the Diaphony and l_N the Logophony respectively of the two-dimensional lifted sequence. Confer (45) we give the following small

table of results:

N, a_2	F_1	F_2	l_N
13,8	1.3952 e-2	6.8982e-1	1.0385
55,34	3.2978 e-2	1.9531e-1	5.8417e-1
377,233	4.8111e-3	3.3745e-2	1.8560e-1
610,377	2.9734e-3	2.1591e-2	1.3401e-1
987,610	1.8376e-3	1.3783e-2	9.5693e-2
1597,987	1.1357e-3	8.7822e-3	6.7675e-3
2584,1597	7.0193e-4	5.5856e-3	4.7457e-3
4181,2584	4.3381e-4	3.5470e-3	3.3033e-3
6765,4181	2.6810e-4	2.2493e-3	2.2841e-3
10946,6765	1.6567e-4	1.4246e-3	1.5699e-3

b) Lifting the sequence $\{ke\}$, $k = 1, \ldots, N$, $e = 2, 71 \ldots$. This classical Kronecker-Sequence has knowingly very good uniform distribution properties. Nevertheless, because the sequence is infinite, by a famous result due to W. Schmid the $D_N^* \geq C \frac{\ln N}{N}$ for infinitely many values of N (confer for example [9]). In the following small table F_1 means the Diaphony of the sequence $\{ke\}$, $k = 1, \ldots, N$ whereas F_2 is the Diaphony of the two-dimensional lifted sequence using the corresponding Fibonacci-Lifting.

N	F_1	F_2
13	2.4019e-1	7.3767e-1
55	6.6385e-2	2.1115e-1
6765	8.2206e-4	2.5034e-3
10946	4.4429e-4	1.5392e-3

It is a consequence of our first theorem, that in example a) the values of F_2 give the exact quality of the Fibonacci-Lifting in terms of the Diaphony of the Lifting. For the values l_2 holds the same, they are independent of the lifted sequences. The same situation occurs in arbitrary dimensions s: Lifting the sequence k/N, $k = 1, \ldots, N$, the resulting Discrepancy D_N^*, the Diaphony F_N and l_N as well give all numerical evidence of the quality of the Lifting which is used.

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