Anzeiger Abt. II (1997) 134: 3–10

Anzeiger
Mathematisch-naturwissenschaftliche Klasse Abt. II
Mathematische, Physikalische und Technische Wissenschaften

Elementary Inequalities in Hypercomplex Numbers

By

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(Vorgelegt in der Sitzung der math.-nat. Klasse am 19. Juni 1997
durch das w. M. Edmund Hlawka)

Abstract

Es werden einige elementare Ungleichungen auf hyperkomplexe Systeme übertragen. Insbesondere werden Ungleichungen für Möbius-Transformationen in diesem allgemeinen Kontext gezeigt.

1. Introduction

In this paper we will extend some known inequalities for complex numbers to certain systems of hypercomplex numbers.

Let $\mathbb{R}^s$ be the Euclidean space of vectors $x = (x_0, x_1, \ldots, x_{s-1}) = x_0e_0 + x_1e_1 + \cdots + x_{s-1}e_{s-1}$. The vectors $e_0, \ldots, e_{s-1}$ denote the standard basis of $\mathbb{R}^s$. Furthermore e_0 is considered to be the real unit $e_0 = 1$ and $e_1, \ldots, e_{s-1}$ are so-called hypercomplex units. x_0 is called real part $\mathrm{Re}(x)$ and $\tilde{x} = \sum_{j=1}^{s-1} x_je_j$ is called the imaginary part $\mathrm{Im}(x)$. The conjugate of x is defined by $\bar{x} = x_0e_0 - \tilde{x}$, and we will further use the notation $\mathrm{Im}_j(x) = x_j$. Let $\langle x, y\rangle$ denote a bilinear product $\mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^s$ such that $\langle e_0, e_j\rangle = e_j$ for $0 \le j \le s-1$, $\langle e_j, e_j\rangle = -e_0$ for $1 \le j \le s-1$ and $\langle e_j, e_k\rangle = -\langle e_k, e_j\rangle$ for $0 \le j < k \le s-1$. In this

* This paper was supported by the "Jubiläumsfonds der Österr. Nationalbank" project 4995.

way $H = (\mathbb{R}^s, +, \langle\cdot,\cdot\rangle)$ becomes an antisymmetric hypercomplex system. One easily sees that $\langle x, \bar{x}\rangle = \langle \bar{x}, x\rangle = \sum_{j=0}^{s-1} x_j^2 = |x|^2$, where $|x|$ is the Euclidean norm of x.

The set $\mathbb{R}$ or $\mathbb{C}$ can be identified with $s = 1$ or $s = 2$, respectively. For $s = 4$ we obtain the quaternion algebra $\mathbb{H}$ provided that $\langle e_1, e_2\rangle = e_3$, $\langle e_2, e_3\rangle = e_1$ and $\langle e_3, e_1\rangle = e_2$. Cayley's octaves $\mathbb{O}$, which are a special case of $s = 8$, can be constructed from $\mathbb{H}$ by the doubling method.

We set $r = |x|, w = |\tilde{x}|$ and $\varphi = \arctan \frac{w}{x_0}$. Defining the powers $x^n = \langle x, x^{n-1}\rangle, n \in \mathbb{N}$, the relations

$$\mathrm{Re}(x^n) = r^n \cos n\varphi, \quad \mathrm{Im}_j(x^n) = r^n \frac{x_j}{w} \sin n\varphi, \tag{1}$$

hold for $1 \le j \le s-1, w \neq 0$, (see [6]). Taking these formulas with $n = 1$ and defining the exponential function with hypercomplex values of x by

$$e^x = \sum_{k\ge 0} \frac{x^k}{k!},$$

we can prove the relation

$$x = r \exp\left(\frac{\tilde{x}}{w}\varphi\right) = r\left(\cos\varphi + \frac{\tilde{x}}{w}\sin\varphi\right).$$

This implies De Moivre's Theorem

$$\left[r\left(\cos\varphi + \frac{\tilde{x}}{w}\sin\varphi\right)\right]^n = r^n\left[\cos(n\varphi) + \frac{\tilde{x}}{w}\sin(n\varphi)\right]$$

for any natural number n.

For $x \in H, x \neq 0$, we define the hypercomplex logarithm of x to be

$$z_n = \log x = \log r + \frac{\tilde{x}}{w}(\varphi + 2n\pi), \quad n \in \mathbb{Z}.$$

Thus, $\log x$ is an infinitely many-valued function; z_0 we call the principal hypercomplex logarithm if $0 \le \varphi < 2\pi$. Furthermore we can introduce logarithmic series, e.g.

$$\log(1+x) = \log|1+x| + \frac{\tilde{x}}{w}\arg(1+x) = \sum_{n\ge 1}(-1)^{n-1}\frac{x^n}{n}, \quad |x| < 1,$$

for which the derivation

$$\frac{d}{dx}\log(1+x) = \frac{1}{1+x} = \sum_{n\ge 0}(-1)^n x^n, |x| < 1$$

holds. We shall need the last two formulas later.

Let n be a positive integer, $a = (a_0, \ldots, a_{s-1})$, $s \geq 2$, a given hypercomplex number and consider the mapping defined by $a \mapsto x$:

$$x^n = a, \quad |x| = |a| = 1. \tag{2}$$

We shall try to find $x = (x_0, \ldots, x_{s-1}) \in H$ which satisfy (2). Using formula (1) we see that

$$a_0 = \mathrm{Re}(x^n) = \cos n\varphi, \tag{3}$$

$$a_j = \mathrm{Im}_j(x^n) = \frac{x_j}{w(x)} \sin n\varphi, 1 \leq j \leq s-1, \text{where} \tag{4}$$

$$\varphi = \arg x = \arctan \frac{w(x)}{x_0}, w(x) \neq 0. \tag{5}$$

With these three equations, we obtain $x_0 = w(x) \cos(\arccos a_0/n)$. Then, taking $w(x) = \sqrt{1 - x_0^2}$ the last relation yields

$$x_0 = \cos\left(\frac{\arccos a_0}{n}\right). \tag{6}$$

Using (3), we can write (4) in the form $x_j/w(x) = a_j/w(a)$. Taking into account $w = \sqrt{1 - x_0^2}$ and (6), this implies

$$x_j = \frac{a_j}{w(a)} \sin\left(\frac{\arccos a_0}{n}\right), \quad 1 \leq j \leq s-1, \quad s \geq 2. \tag{7}$$

We thus have

Proposition 1.1. *Let a and x be hypercomplex numbers of Euclidean norm 1 and let $n \in \mathbb{N}$. Then for any given $a \in H$ there exists a unique $x = (x_0, \ldots, x_{s-1}) \in H$ according (6) and (7) such that $x^n = a$ (We always take the principle values of the trigonometric functions).*

2. A Special Analytic Inequality

Now we want to extend some specific inequalities to the hypercomplex numbers $x = (x_0, \ldots, x_{s-1}) \in H$, $s \geq 1$.

Proposition 2.1. *Let x be a hypercomplex number such that $|x| \leq 1$, then*

$$\frac{|x|}{1 + |x|} \leq |\log(1 + x)| \leq |x| \frac{1 + |x|}{|1 + x|}.$$

Proof: The right-hand inequality can be shown as for complex numbers. For proving the left-hand inequality we take an arbitrary but fixed point x

on the hyper sphere $|x+1| = C, 0 < C < 2, C \in \mathbb{R}$, which extends from the real axis to the unit sphere. The function $|x|/(1+|x|)$ as well as $|\log(1+x)|$ (which is the principal value of the hypercomplex logarithm) are continuous along $|x+1| = C$, and differentiable for $|x+1| < C$. By introducing $\Phi = \arg(x+1)$ as independent variable, the cosine rule yields

$$|x|^2 = |x+1|^2 + 1 - 2|x+1|\cos\Phi \text{ resp. } |x|\frac{d|x|}{d\Phi} = |1+x|\sin|\Phi.$$

The function

$$\frac{d}{d\Phi}\left[|\log(1+x)|^2 - \left(\frac{|x|}{1+|x|}\right)^2\right]$$
$$= 2\Phi - 2\frac{|x|}{(1+|x|)^3}\frac{d|x|}{d\Phi} = 2\Phi - 2\frac{|1+x|}{1+|x|^3}\sin\Phi$$

does not vanish for $\Phi = 0$. This is a contradiction because Rolle's theorem implies that the derivation must vanish somewhere in the interior. From this the result follows immediately.

3. Möbius-Transformations in Hypercomplex Number Systems

In this section we will study four types of Möbius-Transformations in quaternions $\mathbb{H}$ or octaves $\mathbb{O}$. We will prove the following theorem.

Theorem 3.1. *Let $|a| < 1$. Then the Möbius-Transformations*

$$T_1(x) = (x-a)(\bar{a}x-1)^{-1}, \quad T_3(x) = (\bar{a}x-1)^{-1}(x-a),$$
$$T_2(x) = (x-a)(x\bar{a}-1)^{-1}, \quad T_4(x) = (x\bar{a}-1)^{-1}(x-a)$$

map the unit ball bijectively onto itself.

Proof: We will prove the statement only for the function $T_1(x)$ in $\mathbb{O}$. All other cases can be shown along the same lines. Let R be the boundary and I be the interior of the unit ball.

(i) T_1 is invertible on $R \cup I$:
Let $|y| \leq 1, |a| < 1$ and $(x-a)(\bar{a}x-1)^{-1} = y$. From this it follows that

$$x - y(\bar{a}x) = a - y.$$

Each number $x \in \mathbb{O}$ can be considered as a vector $\vec{x} \in \mathbb{R}^8$. Let b be a fixed number. Then the functions $x \to bx$ and $x \to xb$ are linear mappings from $\mathbb{R}^8$ to $\mathbb{R}^8$. We will denote by $[b]_l, [b]_r \in \mathbb{R}^{8\times 8}$ the corresponding matrices. Using this matrix notation, the last equation can be written as

$$(E - [y]_l[\bar{a}]_l)x = a - y,$$

where E denotes the unit matrix. This equation can be solved uniquely, if and only if $\det(E - [y]_l[\bar{a}]_l) \neq 0$, i.e. if there is no eigenvalue 1 of $[y]_l[\bar{a}]_l$.
Consider that 1 is an eigenvalue and v be the corresponding eigenvector. From this follows $[y]_l[\bar{a}]_l v = v$, thus $y(\bar{a}v) = v$. We obtain $|y| \cdot |\bar{a}| = 1$, which is a contradiction to $|y| \leq 1, |a| < 1$.

(ii) T_1 maps R onto R:
Let $|x| = 1$. This implies

$$|T_1(x)| = \left|\frac{x - a}{\bar{a}x - 1}\right| = \left|\frac{x - a}{\bar{a} - \bar{x}}\right| \frac{1}{|x|} = 1,$$

since $x^{-1} = \bar{x}$ for all $x \in R$.

(iii) T_1^{-1} maps R onto R:
From $|y| = 1$ follows

$$|x - a|^2 = |\bar{a}x - 1|^2.$$

Thus

$$(x - a)(\bar{x} - \bar{a}) = (\bar{a}x - 1)(\bar{x}a - 1).$$

A simple computation verifies the identity

$$\bar{a}x - a\bar{x} + \bar{x}a - x\bar{a} = 0 \tag{8}$$

for all x and a. Thus we obtain

$$(1 - |x|^2)(1 - |a|^2) = 0.$$

The second factor of this product is $\neq 0$, thus $|x| = 1$.

(iv) T_1 maps I into I: From (8) we derive

$$1 - \frac{|x - a|^2}{|\bar{a}x - 1|^2} = \frac{(1 - |a|^2)(1 - |x|^2)}{|\bar{a}x - 1|^2}$$

and

$$1 - \left(\frac{|x| \pm |a|}{1 \pm |a||x|}\right)^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{(1 \pm |a||x|)^2}.$$

In the last equation either the upper or the lower sign is true. Applying the triangle inequality

$$1-|a||x| \le |1-\bar{a}x| \le 1+|a||x|$$

to the upper relations we obtain

$$1-\left(\frac{|x|-|a|}{1-|x||a|}\right)^2 \ge 1-|T_1(x)|^2 \ge 1-\left(\frac{|x|+|a|}{1+|x||a|}\right)^2.$$

From this we derive

$$\left(\frac{|x|-|a|}{1-|x||a|}\right)^2 \le |T_1(x)|^2 \le \left(\frac{|x|+|a|}{1+|x||a|}\right)^2 < 1.$$

(v) T_1^{-1} maps I into I:
Consider, there is an $x = T_1^{-1}(y)$ with $y \in I, x \notin I$. Let $y_0 = 0 \in I$, then we have $x_0 = T^{-1}(y_0) = a \in I$. We consider the straight line $\overline{y_0 y} \subset I$. Since T_1^{-1} is continuous, there must exist a $\tilde{y} \in \overline{y_0 y}$ with $T_1^{-1}(\tilde{y}) = \tilde{x} \in R$. From this we obtain $\tilde{y} = T_1(\tilde{x}) \in R$, which is a contradiction to $\tilde{y} \in \overline{y_0 y} \subset I$.

From the steps (i)–(v) we conclude the theorem. □

4. Further Analytic Inequalities

Recall that we have introduced the exponential function by the power series which converges absolutely for all $x \in \mathbb{H}$. An alternative definition of the exponential function is

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

It can be expanded by the binomial theorem and the convergence proof can be carried through as in the usual case.

For a natural number n and any hypercomplex number $x \neq 0$ the following result can be shown by induction:

$$\left|e^x - \left(1+\frac{x}{n}\right)^n\right| < \left|e^{|x|} - \left(1+\frac{|x|}{n}\right)^n\right| < e^{|x|}\frac{|x|^2}{2n}.$$

Proposition 4.1. *Suppose that $a_n \in H$ with $a_n = \mathbf{O}(\frac{1}{n!})$. Then the estimate*

$$\left|\sum_{k=0}^{\infty}\langle a_k, x^k\rangle - (a_0 + \langle a_1, x\rangle + \cdots + \langle a_n, x^n\rangle)\right| \le N|x|^{n+1}e^{|x|}$$

holds for any natural number n and $x \in H$.

Proof: We have

$$\left|\sum_{k=0}^{\infty}\langle a_k, x^k\rangle - (a_0 + \langle a_1, x\rangle + \cdots + \langle a_n, x^n\rangle)\right| = \left|\sum_{k=n+1}^{\infty} a_k x^k\right|$$
$$\leq M_1 \frac{|x|^{n+1}}{(n+1)!}\sum_{k=0}^{\infty}\frac{|x|^k}{k!} = M|x|^{n+1}e^{|x|}, \quad M \in \mathbb{R}^+.$$

In the following we list a few inequalities for elementary functions in hypercomplex variables $x = x_0 e_0 + \tilde{x} \in H$. We use the definition of *hypercomplex sine* and *cosine* functions by Taylor series or by the generalized Euler formula

$$e^{(\frac{\tilde{x}}{w}\varphi)} = \cos\varphi + \frac{\tilde{x}}{w}\sin\varphi,$$

which links the exponential function with the trigonometric functions:

$$\sin x = \sum_{k\geq 0}\frac{(-1)^k}{(2k+1)!}x^{2k+1}, \quad \cos x = \sum_{k\geq 0}\frac{(-1)^k}{(2k)!}x^{2k},$$
$$\sin x = -\frac{\tilde{x}}{2w}(e^{\frac{\tilde{x}}{w}x} e^{-\frac{\tilde{x}}{w}x}), \quad \cos x = \frac{1}{2}(e^{\frac{\tilde{x}}{w}x} + e^{-\frac{\tilde{x}}{w}x}).$$

Similarly, the *hypercomplex hyperbolic* functions are given by

$$\sinh x = \sum_{k\geq 0}\frac{x^{2k+1}}{(2k+1)!}, \quad \cosh x = \sum_{k\geq 0}\frac{x^{2k}}{(2k)!},$$
$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x}).$$

With these obvious definitions most of the familiar real- and complex valued trigonometric resp. hyperbolic inequalities can be extended to the hypercomplex system H (For details we refer to [5]).

Proposition 4.2. *If* $|x| < 1$, *then*

$$|\sin x| \leq \frac{|x|}{2}\left(e - \frac{1}{e}\right), \quad |\cos x| < \frac{1}{2}\left(e + \frac{1}{e}\right),$$
$$|\sinh x| \leq \frac{|x|}{2}\left(e - \frac{1}{e}\right), \quad |\cosh x| < \frac{1}{2}\left(e + \frac{1}{e}\right),$$
$$\text{and} \quad (3-e)|x| \leq |e^x - 1| \leq (e-1)|x|.$$

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