

# A Pexider-Type Equation in Normed Linear Spaces

By

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(Vorgelegt in der Sitzung der math.-nat. Klasse am 16. Oktober 1997  
durch das w. M. Ludwig Reich)

## Abstract

Answering a question posed recently by Ludwig Reich, we give a description of solutions to the functional equation

$$\|f(x+y)\| = \|g(x) + b(y)\|.$$

Unexpectedly, even in strictly convex spaces, this equation fails to be equivalent to the Pexider functional equation.

## 1. Introduction

The aim of this paper is to answer a question posed by Professor Ludwig Reich during my stay at the Karl-Franzens Universität (Graz, Austria, Autumn 1994): give a description of solutions to the functional equation

$$\|f(x+y)\| = \|g(x) + b2l(y)\|. \quad (1)$$

The case where  $f = g = b$  has extensively been studied by many authors, see e.g. P. Fischer and G. Muszély [5], J. Dhombres [3], J. Aczél and J. Dhombres [2] and R. Ger [6]. The reason why the functional equation

$$\|f(x+y)\| = \|f(x) + f2l(y)\| \quad (F)$$

aroused so much interest is, on one side, the fact that it generalizes

conditional Cauchy functional equations of the type

$$f(x + y)^n = (f(x) + f(y))^n$$

(widely investigated in sixties) and, on the other side, because of its links with the theory of isometries; last but not least it leads to some characterizations of strictly convex normed linear spaces as well as to some of their generalizations. The main result from [6] states that any map  $f$  from a (not necessarily commutative) group into a strictly convex space has to be additive, i.e. to satisfy the Cauchy equation

$$f(x + y) = f(x) + f(2l(y)). \tag{C}$$

Surprisingly, in contrast to that, even in the case of strictly convex ranges, equation (1) fails to be equivalent to the Pexider functional equation

$$f(x + y) = g(x) + b2l(y). \tag{P}$$

Indeed, let  $(X, +)$  be a groupoid and let  $(Y, \| \cdot \|)$  be a normed linear space with  $\dim Y \geq 2$ . Fix arbitrarily a nonnegative number  $\varrho$  and a  $d \in Y$ . Denoting by  $S(a, \varrho)$  the sphere  $\{u \in Y : \|u - a\| = \varrho\}$ ,  $a \in Y$ , one can easily check that the triple  $(f, g, d)$  yields a solution to (1) for *quite arbitrary* mappings  $f : X \rightarrow S(0, \varrho)$  and  $g : X \rightarrow S(-d, \varrho)$ . Therefore, in general, equation (1) enjoys an abundance of solutions being far from translations of an additive map which are the only ones satisfying the Pexider equation (cf. J. Aczél [1] or M. Kuczma [7], for instance). As we shall see later on such a phenomenon is caused by the lack of zeros of the map  $f$ . If  $f$  vanishes at at least one point of its domain then all the triples  $(f, g, b)$  fulfilling (1) may be expressed in terms of mappings  $G$  fulfilling the equation

$$\|G(x - y)\| = \|G(x) - G2l(y)\|. \tag{2}$$

We terminate this paper with a detailed discussion of solutions of this equation.

## 2. Solutions Admitting Zeros

Assuming that either  $f$  or, equivalently, the two-place function  $(x, y) \rightarrow g(x) + b2l(y)$  vanishes at some point we shall reduce equation (1) to (2). Namely we have the following

**Theorem 1.** *Let  $(X, +)$  be a group (not necessarily commutative) and let  $(Y, \| \cdot \|)$  be a (real or complex) normed linear space. Assume that functions  $f, g, b: X \rightarrow Y$  satisfy the functional equation (1) for all  $x, y \in X$  and  $f(x_0) = 0$  for some*

$x_0 \in X$ . Then there exists a solution  $G : X \rightarrow Y$  of equation (2) and a vector  $a \in Y$  such that

$$g(x) = G(x) + a, \quad x \in X, \tag{3}$$

$$b(x) = -G(x_0 - x) - a, \quad x \in X, \tag{4}$$

and  $f$  is a selection of the multifunction

$$X \ni x \rightarrow S(0, \|G(x) - G(x_0)\|) \subset Y. \tag{5}$$

Conversely, for every solution  $G : X \rightarrow Y$  of equation (2), for every vector  $a \in Y$ , for every point  $x_0 \in X$  and for every selection  $f$  of the multifunction (5), the triple  $(f, g, b)$  with  $g$  and  $b$  given by (3) and (4), respectively, yields a solution to (1) with  $f(x_0) = 0$ .

*Proof:* Assume that functions  $f, g, b : X \rightarrow Y$  satisfy (1) and  $f(x_0) = 0$ . Since, for every  $t \in X$  we have

$$0 = \|f(x_0)\| = \|f((x_0 - t) + t)\| = \|g(x_0 - t) + b(t)\|$$

we infer that

$$b(t) = -g(x_0 - t) \quad \text{for all } t \in X.$$

Consequently, for all  $x, y \in X$  one has

$$\|f(x + y)\| = \|g(x) - g(x_0 - y)\|,$$

or, equivalently,

$$\|f(x - y + x_0)\| = \|g(x) - g(2I(y))\|.$$

Setting  $F(t) := f(t + x_0)$  and  $G(t) := g(t) - g(0)$ ,  $t \in X$ , we get the relationship

$$\|F(x - y)\| = \|G(x) - G(2I(y))\|,$$

valid for every  $x, y \in X$ . In particular, since obviously  $G(0) = 0$ , the equality  $\|F(x)\| = \|G(x)\|$  holds true for all  $x \in X$ , which implies that

$$\|G(x - y)\| = \|G(x) - G(2I(y))\| \quad x, y \in X,$$

i.e. equation (2) is satisfied. Putting  $a := g(0)$  we get (4) by means of the definition of  $G$  and  $b(x) = -g(x_0 - x) = -G(x_0 - x) - a$  for all  $x \in X$ , which proves the validity of (5). To finish the necessity part of the proof observe that

$$\|f(x)\| = \|F(x - x_0)\| = \|G(x - x_0)\| = \|G(x) - G(x_0)\|, \quad x \in X,$$

which says that

$$f(x) \in S(0, \|G(x) - G(x_0)\|)$$

for all  $x \in X$ , as claimed.

To prove the converse, fix arbitrarily  $x, y \in X$ . Then applying relationships (5), (2), (3) and (4) subsequently, we arrive at

$$\begin{aligned} & \|f(x+y)\| \\ &= \|G(x+y) - G(x_0)\| = \|G(x+y-x_0)\| = \|G(x - (x_0 - y))\| \\ &= \|G(x) - G(x_0 - y)\| + \|(G(x) + a) + (-G(x_0 - y) - a)\| \\ &= \|g(x) + b2l(y)\|, \end{aligned}$$

which completes the proof.

**Remark 1.** The assumption on  $f$  to possess a zero in  $X$  may equivalently be replaced by the requirement

$$b^{-1}(-g(X)) \neq \emptyset \quad \text{or} \quad g^{-1}(-b(X)) \neq \emptyset.$$

In particular, this is the case provided that at least one of the maps  $g$  and  $b$  is *surjective*.

*Proof:* If, say  $\nu \in b^{-1}(-g(X))$ , then there exists a  $\nu \in X$  such that  $b(\nu) = -g(u)$  whence by setting  $x_0 := u + \nu$  we get

$$\|f(x_0)\| = \|f(u + \nu)\| = \|g(u) + b(\nu)\| = 0.$$

**Theorem 2.** *Under the assumptions of Theorem 1 if, additionally, the target space  $(Y, \|\cdot\|)$  is strictly convex, functions  $f, g, b : X \rightarrow Y$  satisfy equation (1),  $f(x_0) = 0$  for some  $x_0 \in X$ , and either the even part of  $g$  is constant or the function  $X \ni x \rightarrow b(x + x_0) \in y$  has constant even part, then there exists an additive map  $G : X \rightarrow Y$  and constants  $a, b \in Y$  such that*

$$g(x) = G(x) + a, \quad x \in X, \tag{3'}$$

$$b(x) = G(x) + b, \quad x \in X, \tag{4'}$$

and  $f$  is a selection of the multifunction

$$X \ni x \mapsto S(0, \|G(x) + a + b\|) \subset Y. \tag{5'}$$

*Conversely, for every additive function  $G : X \rightarrow Y$ , for every vectors  $a, b \in Y$  and for every selection  $f$  of the multifunction (5'), the triple  $(f, g, b)$  with  $g$  and  $b$  given by (3') and (4'), respectively, yields a solution to (1).*

*Proof:* Only the “if” part requires a motivation. On account of Theorem 1, there exists a solution  $G : X \rightarrow Y$  of equation (2) and a vector  $a \in Y$  such that relationships (3), (4) and (5) hold true.

Assume first that the even part of  $g$  is constant. Since each solution  $G$  of (2) vanishes at zero we get  $\text{const} \equiv g(x) + g(-x) = G(x) + G(-x) + 2a = 2a$  which proves that  $G$  is odd. Consequently, equation (2) assumes the form

$$\| G(x + y) \| = \| G(x) + G(y) \|$$

for all  $x, y \in X$ . An appeal to [6, Theorem 1] shows that  $G$  has to be additive. Therefore relation (4) reduces itself to (4') with  $b := -G(x_0) - a$  and, similarly, the multifunction (5) is transformed into (5').

Assuming now that the even part of the function  $X \ni x \mapsto b(x + x_0) \in Y$  is constant we have to have

$$-2a = 2b(x_0) \equiv b(x + x_0) + b(-x + x_0) = -2a - [G(x) + G(-x)]$$

which, again, shows that  $G$  is odd. Repeating the last part of the previous considerations completes the proof.

**Remark 2.** A particular selection

$$f(x) := G(x) + a + b, \quad x \in X,$$

of the multifunction (5') in Theorem 2 leads to a solution  $(f, g, b)$  of the Pexider equation (P). However, in general, Theorem 2 shows that even in the case of strictly convex ranges, a solution  $(f, g, b)$  of (1) may still be far from any triple solving (P) because of multitude of possible selections  $f$ . Nevertheless, remarkable is the fact that functions  $g$  and  $b$  in any such triple are exactly those occurring in solutions of the Pexider equation (translations of an additive function).

### 3. Basic Equation and Additivity

Equation (2), it appears, happened to be basic while studying (1). Obviously, each *odd* solution of (2) satisfies (F) and every solution of (F) is easily checked to be odd. Therefore

**Remark 3.** Equations (2) and (F) are equivalent in the class of odd functions mapping a group into a normed linear space.

Replacing  $x$  by  $x + y$  in (2) we arrive at

$$\| G(x) \| = \| G(x + y) - G2I(y) \|,$$

which, in case of Abelian domains, is equivalent to

$$\| G(x + y) - G(x) \| = \| G2I(y) \| . \tag{S}$$

Equally simple is the way back whence

**Remark 4.** Equations (2) and (S) are equivalent in the class of functions mapping a commutative group into a normed linear space.

Equation (S) was examined by F. Skof [8] in the case where the unknown function  $G$  is defined on a real linear space. Her principal goal was to give sufficient conditions for a solution of (S) to be additive. As we shall see later on, the main results ([8, Theorems 1 and 2]) are special cases of our Theorem 3 (ii) and Corollary 2, respectively.

We proceed with the following

**Theorem 3.** *Let  $X, +$  be an Abelian group and let  $(Y, \| \cdot \|)$  be a strictly convex normed linear space. If  $g: X \mapsto Y$  is a solution to equation*

$$\| G(x - y) \| = \| G(x) - G(y) \| , \quad x, y \in X, \tag{2}$$

*then the following conditions are pairwise equivalent:*

- (i)  $G$  is additive;
- (ii)  $G(X) = -G(X)$ ;
- (iii)  $G$  is odd;
- (iv)  $\| G(2x) \| = 2 \| G(x) \|$  for all  $x \in X$ .

*Proof:* Implication (i)  $\Rightarrow$  (ii) results from the fact that any additive function is odd.

To prove that (ii)  $\Rightarrow$  (iii) fix arbitrarily an  $x \in X$  and choose a  $y \in X$  to have  $G(y) = -G(x)$ . Then, apply relation (2) twice to get

$$\| G(x + y) + G(x) \| = \| G(x) \| = \| G(x) - G(x + y) \| \tag{6}$$

implying that

$$\begin{aligned} & \| G(x + y) + G(x) \| + \| G(x) - G(x + y) \| \\ &= 2 \| G(x) \| = \| (G(x + y) + G(x)) + (G(x) - G(x + y)) \| . \end{aligned}$$

Now, on account of the strict convexity of  $Y$  we derive the existence of a nonnegative scalar  $\lambda(x, y)$  such that

$$G(x + y) + G(x) = \lambda(x, y)(G(x) - G(x + y)).$$

By means of (6) this shows that  $\lambda(x, y) = 1$  whenever  $G(x) \neq 0$  and,

consequently,

$$0 = \|G(y + x)\| = \|G(y - (-x))\| = \|G(y) - G(-x)\| \\ = \|G(x) + G(-x)\|.$$

Thus  $G(-x) = -G(x)$  which, obviously, holds true also in the case where  $G(x) = 0$  (put  $y = -x$  in (S)).

Assuming (iii) and setting  $y = -x$  in (2) we get (iv).

The task is now to show that (iv)  $\Rightarrow$  (i). Replacing  $y$  by  $-y$  in (2) we infer that

$$\|G(x + y)\| = \|G(x) - G(-y)\|, \quad x, y \in X. \quad (7)$$

In particular,  $\|G(y)\| = \|G(-y)\|, y \in X$ , for  $G(0) = 0$  whence, for every  $x \in X$ , one has

$$\|G(x)\| + \|-G(-x)\| = 2\|G(x)\| = \|G(2x)\| \\ = \|G(x) + (-G(-x))\|$$

because of (6) applied for  $y = x$ . Now, by means of the strict convexity of  $Y$ , for every  $x \in X$  there exists a nonnegative number  $\lambda(x)$  such that  $-G(-x) = \lambda(x)G(x)$ . Therefore, an appeal to (7) shows that

$$\|G(x + y)\| = \|G(x) + \lambda 2I(y)G2I(y)\|, \quad x, y \in X. \quad (8)$$

Putting here  $x = 0$  gives the equality

$$\|G2I(y)\| = \lambda 2I(y)\|G2I(y)\|$$

valid for every  $y \in Y$ , which implies that  $\lambda 2I(y) = 1$  provided that  $G(y) \neq 0$ . Consequently, by virtue of (8), for all  $x, y \in X$ , we get

$$\|G(x + y)\| = \|G(x) + G2I(y)\|$$

whenever  $G(y) \neq 0$ . However, the latter equation is satisfied also in the case where  $G(y) = 0$ , since then  $G(-y) = 0$  as well and (2) proves that  $\|G(x + y)\| = \|G(x) + G(-y)\| = \|G(x)\|$  for all  $x \in X$ . Thus,  $G$  satisfies equation (F) and it remains to apply Theorem 1 from [6] once again to show that  $G$  is also a solution to the Cauchy equation (C). This finishes the proof.

**Remark 5.** The commutativity of  $(X, +)$  was used *exclusively* to show that (ii)  $\Rightarrow$  (iii). Even in this case the relationship

$$\|G(x + y)\| = \|G(y + x)\|, \quad x, y \in X, \quad (9)$$

is sufficient to conduct the part of the proof of Theorem 3. Indeed, having (9) we replace  $y$  by  $y - x$  to get

$$\|G(y)\| = \|G(x + y - x)\| = \|G(x + y) - G(x)\|$$

and that is what was really needed. The question whether or not equation (2) implies (9) in non-Abelian groups remains open.

**Remark 6.** Unlike (F) equation (2) always admits nonadditive solutions (no matter whether or not the target space is strictly convex) provided that the domain constitutes a group possessing subgroups of index 2. If that is the case,  $(K, +)$  is a subgroup of index 2 of the group  $(X, +)$  and  $c \neq 0$  is an arbitrarily fixed vector of the normed linear space  $(Y, \|\cdot\|)$ , then any function  $G : X \rightarrow Y$  given by the formula

$$G(x) = \begin{cases} 0 & \text{if } x \in K \\ c & \text{if } x \in X \setminus K \end{cases} \tag{10}$$

yields a nonadditive solution of equation (2). Indeed,  $G$  being even and nonzero cannot be additive since, otherwise, it would be odd. To check that it satisfies equation (2) fix arbitrarily a pair  $(x, y) \in X^2$ . The following three possibilities have to be distinguished:

- (a)  $x, y \in K$ : then so does  $x - y$  and both sides of (2) are equal to 0;
- (b)  $x, y \in X \setminus K$ : then  $x - y$  is in  $K$  and we have the equalities

$$G(x - y) = 0 = c - c = G(x) - G(y);$$

(c) exactly one of the arguments  $x, y$  is in  $K$ : then  $x - y \in X \setminus K$  whence  $G(x - y) = c$  and  $G(x) - G(y) \in \{-c, c\}$ ; thus (2) is satisfied as well.

**Remark 7.** Functions of the form (10) are, jointly with the additive solutions, the only ones that satisfy Mikusiński's functional equation

$$G(x + y) \neq 0 \text{ implies } G(x + y) = G(x) + G(y). \tag{M}$$

(cf. L. Dubikajtis, C. Ferens, R. Ger & M. Kuczma [4] or M. Kuczma [7]). Therefore, in the light of Remark 6, each solution of equation (M) satisfies the basic equation (2). In the next section we shall show, among others, that the converse is true in the case of real functionals on groups.

#### 4. Solutions with Values in Inner-Product Spaces

Except for Theorem 4 below, in the present section we deal with solutions to the basic equation (2) which map a given group into an inner-product space. So, we replace the assumption of strict convexity upon the target space by a *stronger* requirement: the norm comes from an inner-product



structure. In particular, every odd solution in the class spoken of is necessarily additive (see Theorem 3). It turns out that under the 2-divisibility assumption upon the domain the even solutions are just the trivial ones regardless of the kind of norm considered.

**Theorem 4.** *Let  $(X, +)$  be a 2-divisible group (not necessarily commutative) and let  $(Y, (\|\cdot\|))$  be a normed linear space (real or complex). Then any even solution of equation (2), mapping  $X$  into  $Y$  vanishes identically on  $X$ .*

*Proof:* Let  $G : X \rightarrow Y$  be an even solution of (2). Replacing  $y$  by  $-y$  in (2) leads to

$$\|G(x + y)\| = \|G(x) - G(y)\|, \quad x, y \in X,$$

whence, by putting here  $y = x$  we obtain the equality

$$G(2x) = 0$$

valid for all  $x \in X$ . An appeal to the 2-divisibility assumption on  $X$  completes the proof.

**Remark 8.** In view of Remark 4 the divisibility assumption is essential because each function of the form (10) is even.

**Theorem 5.** *Let  $(X, +)$  be an arbitrary group (not necessarily commutative) and let  $(Y, (\cdot|\cdot))$  be an inner-product space (real or complex). Then  $G : X \rightarrow Y$  is a solution of the equation*

$$\|G(x - y)\| = \|G(x) - G_e 2I(y)\|, \quad x, y \in X, \quad (2)$$

*if and only if*

$$\|G(x) + G_e 2I(y)\|^2 = \|G(x + y)\|^2 + 4 \operatorname{Re}(G(x)|G_e 2I(y)) \quad (11)$$

*for all,  $x, y \in X$ , where  $G_e$  stands for the even part of  $G$ .*

*Proof:* Let  $G : X \rightarrow Y$  be a solution of (2). Fix arbitrarily  $x, y \in X$ . Then

$$\begin{aligned} \|G(x + y)\|^2 &= \|G(x) - G(-y)\|^2 = \|G(x)\|^2 \\ &\quad + \|G_e 2I(y)\|^2 - 2 \operatorname{Re}(G(x)|G(-y)) \end{aligned}$$

and, similarly,

$$\begin{aligned} \|G(x - y)\|^2 &= \|G(x) - G_e 2I(y)\|^2 = \|G(x)\|^2 \\ &\quad + \|G_e 2I(y)\|^2 - 2 \operatorname{Re}(G(x)|G_e 2I(y)) \end{aligned}$$

whence, by means of the parallelogram property of the norm

$$\begin{aligned} \|G(x+y)\|^2 + \|G(x-y)\|^2 &= 2\|G(x)\|^2 + 2\|G2I(y)\|^2 \\ &\quad - 4\operatorname{Re}(G(x)|G_e2I(y)) \\ &= \|G(x) + G2I(y)\|^2 + \|G(x) - G2I(y)\|^2 - 4\operatorname{Re}(G(x)|G_e2I(y)). \end{aligned}$$

Now, applying equation (2) once again, we get (11).

Conversely, assume the validity of (11) for all  $x, y \in X$ . Observe first that  $G(0) = 0$  which can easily be derived by setting  $x = y = 0$  in (11). Now, setting  $y = -x$  in (11), we obtain the equalities

$$\begin{aligned} \|G(x) + G(-x)\|^2 &= 4\operatorname{Re}(G(x)|G_e(x)) = 2\|G(x)\|^2 \\ &\quad + 2\operatorname{Re}(G(x)|G(-x)) \end{aligned}$$

and, consequently,

$$\|G(-x)\| = \|G(x)\|, \quad x \in X.$$

Equation (11) says now that

$$\|G(x)\|^2 - 2\operatorname{Re}(G(x)|G(-y)) + \|G(-y)\|^2 = \|G(x+y)\|^2$$

or, equivalently,

$$\|G(x) - G(-y)\| = \|G(x - (-y))\|$$

holds true for all  $x, y \in X$ , which, evidently, says nothing else but (2) and the proof has been completed.

**Theorem 6.** *Let  $(X, +)$  be a commutative group and let  $(Y, (\cdot|\cdot))$  be a real inner-product space. Then equation (2) is equivalent to the system*

$$\begin{aligned} \|G(x) + G2I(y)\|^2 &= \|G(x+y)\|^2 + \|G(x) + G2I(y) - G(x+y)\|^2 \\ \|G(x) + G2I(y) - G(x+y)\|^2 &= 4(G(x)|G_e2I(y)) \end{aligned}$$

assumed for all  $x, y \in X$ .

*Proof:* Let  $G : X \rightarrow Y$  be a solution of (2). With the aid of Theorem 5 we get (11) and, therefore, it suffices to check the validity of the first equation of the system considered. To this end, note that for every  $x, y \in X$  one has

$$\begin{aligned} \|G2I(y)\|^2 &= \|G(x) - G(x-y)\|^2 = \|G(x)\|^2 \\ &\quad - 2(G(x)|G(x-y)) + \|G(x-y)\|^2 \\ &= 2\|G(x)\|^2 - 2(G(x)|G(x-y) + G2I(y)) + \|G2I(y)\|^2, \end{aligned}$$

i.e.

$$\| G(x) \|^2 = (G(x)|G(x - y) + G2I(y)).$$

This clearly forces

$$\begin{aligned} 0 &= (G(x)|G(x - y) + G2I(y) - G(x)) \\ &= \frac{1}{4} [\| G(x - y) + G2I(y) \|^2 - \| 2G(x) - G(x - y) - G2I(y) \|^2] \end{aligned}$$

and, in consequence, the equality

$$\| G(x - y) + G2I(y) \| = \| 2G(x) - G(x - y) - G2I(y) \|$$

is satisfied for all  $x, y \in X$ . Finally, replacing here  $x$  by  $x + y$ , with the aid of the parallelogram property, we conclude that

$$\begin{aligned} \| G(x) + G2I(y) \|^2 &= \| 2G(x + y) - G(x) - G2I(y) \|^2 \\ &= \| G(x + y) + (G(x + y) - G(x) - G2I(y)) \|^2 \\ &= 2 \| G(x + y) \|^2 + 2 \| G(x + y) - G(x) - G(y) \|^2 \\ &\quad - \| G(x) + G2I(y) \|^2 \end{aligned}$$

holds true for every  $x, y \in X$ , which is our claim.

Conversely, assuming that the system is satisfied for all pairs of arguments  $(x, y) \in X^2$ , we immediately derive the relationship (11) which, by virtue of Theorem 5, implies the validity of equation (2). This ends the proof.

**Corollary 1.** *Under the assumptions of Theorem 6 every solution  $G : X \rightarrow Y$  of equation (2) has the following property:*

$$G_o(x) \perp G_e2I(y) \quad \text{for every } x, y \in X.$$

where  $G_o$  and  $G_e$  stand for the odd and even part of  $G$ , respectively. In particular, if the set  $\{G_o(x) : x \in X\}$  is total then  $G$  is additive.

*Proof:*

$$\| G(x) + G2I(y) \|^2 = \| G(x + y) \|^2 + 4(G(x)|G_e2I(y))$$

for all  $x, y \in X$ . Interchanging the roles of  $x$  and  $y$ , due to the commutativity of  $(X, +)$  we get also

$$\| G(x) + G2I(y) \|^2 = \| G(x + y) \|^2 + 4(G2I(y)|G_e(x))$$

whence

$$(G(x)|G_e2I(y) = (G2I(y)|G_e(x))$$

for all  $x, y \in X$ . Plainly, we have

$$(G(-x)|G_e 2l(y) = (G 2l(y)|G_e(x)), \quad x, y \in X,$$

which, by subtracting these two equalities, leads to the desired conclusion.

Recall that a subset  $\mathcal{A}$  of an inner product space  $(Y, (\cdot|\cdot))$  is termed *total* provided that the zero vector in  $Y$  is the only one being perpendicular to every member of the set  $\mathcal{A}$ . Therefore, assuming that the set  $\{G_o(x) : x \in X\}$  is total we infer that  $G_e$  has to vanish identically on  $X$ . Thus  $G$  itself is an odd function and it remains to apply Theorem 3 (iii) to finish the proof.

**Theorem 7.** *Let  $(X, +)$  be a commutative group. Then a function  $G : X \rightarrow \mathbb{R}$  satisfies the equation*

$$|G(x-y)| = |G(x) - G 2l(y)|, \quad x, y \in X, \quad (12)$$

*if and only if  $G$  is a solution to Mikusiński's equation*

$$G(x+y)[G(x+y) - G(x) - G 2l(y)] = 0, \quad x, y \in X. \quad (13)$$

*Proof:* Let  $G : X \rightarrow \mathbb{R}$  be a solution of (12). An appeal to Theorem 6 shows that

$$G(x+y) \perp G(x+y) - G(x) - G 2l(y)$$

for all  $x, y \in X$  which, in the real case, states nothing else but (13).

As to the converse Remark 7 may directly be applied. This ends the proof.

Remark 7 jointly with Theorem 7 immediately imply the following

**Corollary 2.** *If  $(X, +)$  is a commutative group with no subgroups of index 2, then a function  $G : X \rightarrow \mathbb{R}$  satisfies equation (12) if and only if  $G$  is additive.*

### Acknowledgement

This research was supported by KBN grant No. 2 P03A 049 09 (Poland).

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