Sitzungsber. Abt. II (1998) 207: 71–82

On Gauss-Pólya's Inequality

By

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(Vorgelegt in der Sitzung der math.-nath. Klasse am 18. Juni 1998
durch das w. M. Peter Gruber)

Abstract

Let $g, h : [a, b] \to \mathbf{R}$ be nonnegative nondecreasing functions such that g and h have a continuous first derivative and $g(a) = h(a), g(b) = h(b)$. Let $p = (p_1, p_2)$ be a pair of positive real numbers p_1, p_2 such that $p_1 + p_2 = 1$.

a) If $f : [a, b] \to \mathbf{R}$ be a nonnegative nondecreasing function, then for $r, s < 1$

$$M_p^{[r]}\left(\int_a^b g'(t) f(t)\, dt, \int_a^b h'(t) f(t)\, dt\right) \leq \int_a^b (M_p^{[s]}(g(t), h(t)))' f(t)\, dt \tag{1}$$

holds, and for $r, s > 1$ the inequality is reversed.

b) If $f : [a, b] \to \mathbf{R}$ is a nonnegative nonincreasing function then for $r < 1 < s$ (1) holds and for $r > 1 > s$ the inequality is reversed. Similar results are derived for quasiarithmetic and logarithmic means.

Key words and phrases: Logarithmic mean, quasiarithmetic mean, Pólya's inequality, weighted mean.
Mathematics Subject Classification (1991): 26D10.

1. Introduction

Gauss mentioned the following result in [2]:

If f is a nonnegative and decreasing function then

$$\left(\int_0^\infty x^2 f(x)\, dx\right)^2 \leq \frac{5}{9} \int_0^\infty f(x)\, dx \int_0^\infty x^4 f(x)\, dx. \tag{2}$$

Pólya and Szegö classical book "Problems and Theorems in Analysis, I" [7] gives the following generalization and extension of Gauss' inequality (2).

Theorem A. (*Pólya's inequality*) *Let a and b be nonnegative real numbers.*
a) If $f\colon [0,\infty) \to \mathbf{R}$ is a nonnegative and decreasing function, then

$$\left(\int_0^\infty x^{a+b} f(x)\,dx\right)^2 \le \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^\infty x^{2a} f(x)\,dx \times \int_0^\infty x^{2b} f(x)\,dx \tag{3}$$

whenever the integrals exist.
b) If $f\colon [0,1) \to \mathbf{R}$ is a nonnegative and increasing function, then

$$\left(\int_0^1 x^{a+b} f(x)\,dx\right)^2 \ge \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a} f(x)\,dx \times \int_0^1 x^{2b} f(x)\,dx. \tag{4}$$

Obviously, putting $a = 0$ and $b = 2$ in (3) we obtain Gauss' inequality. Recently Pečarić and Varošanec [6] obtained a generalization.

Theorem B. *Let $f : [a,b] \to \mathbf{R}$ be nonnegative and increasing, and let $x_i : [a,b] \to \mathbf{R}(i = 1,\dots,n)$ be nonnegative increasing functions with a continuous first derivative. If $p_i, (i = 1,\dots,n)$ are positive real numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$, then*

$$\int_a^b \left(\prod_{i=1}^n (x_i(t))^{1/p_i}\right)' f(t)\,dt \ge \prod_{i=1}^n \left(\int_a^b x_i'(t) f(t)\,dt\right)^{1/p_i}. \tag{5}$$

If $x_i(a) = 0$ for all $i = 1,\dots,n$ and if f is a decreasing function then the reverse inequality holds.

The previous result is an extension of the Pólya's inequality. If we substitute in (5): $n = 2, p_1 = p_2 = 2, a = 0, b = 1, g(x) = x^{2u+1}, h(x) = x^{2v+1}$ where $u, v > 0$, we have (4).

In this paper we provide generalizations of Theorem B in a number of directions. In Section 2 we first provide the inequality for weighted means. We note that, as is suggested by notation for means, our result extends to the case when the ordered pair (p_1, p_2) is replaced by an n-tuple. We derive also a version of our theorem for higher derivatives.

Section 4 treats some corresponding results when M is replaced by quasiarithmetic mean. This can be done when the function involved enjoys appropriate convexity properties. A second theorem in Section 4 allows one weight p_1 to be positive and the others negative.

Section 5 addresses the logarithmic mean.

2. Results Connected with Weighted Means

$M_p^{[s]}(a)$ denotes the weighted mean of order r and weights $p = (p_1, \ldots, p_n)$ of a positive sequence $a = (a_1, \ldots, a_n)$. The n-tuple p is of positive numbers p_i with $\sum_{1=i}^{n} p_i = 1$. The mean is defined by

$$M_p^{[r]}(a) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} p_i a_i^r \right)^{1/r} & \text{for} \quad r \neq 0 \\ \displaystyle\prod_{i=1}^{n} a_i^{p_i} & \text{for} \quad r = 0. \end{cases}$$

In the special cases $r = -1, 0, 1$ we obtain respectively the familiar harmonic, geometric and arithmetic mean.

The following theorem, which is a simple consequence of Jensen's inequality for convex functions, is one of the most important inequalities between means.

Theorem C. *If a and p are positive n-tuples and* $s < t$, $s, t \in \mathbf{R}$, *then*

$$M_p^{[s]}(a) \leq M_p^{[t]}(a) \quad \textit{for} \quad s < t, \tag{6}$$

with equality if and only if $a_1 = \ldots = a_n$.

A well-known consequence of the above statement is the inequality between arithmetic and geometric means. Previous results and refinements can be found in [3].

The following theorem is the generalization of Theorem B.

Theorem 1. *Let* g, $h : [a, b] \to \mathbf{R}$ *be nonnegative nondecreasing functions such that* g *and* h *have a continuous first derivative and* $g(a) = h(a), g(b) = h(b)$. *Let* $p = (p_1, p_2)$ *be a pair of positive real numbers* p_1, p_2 *such that* $p_1 + p_2 = 1$.

a) If $f : [a, b] \to \mathbf{R}$ *be a nonnegative nondecreasing function, then for* $r, s < 1$

$$M_p^{[r]}\left(\int_a^b g'(t) f(t)\, dt, \int_a^b h'(t) f(t)\, dt \right) \leq \int_a^b \left(M_p^{[s]}(g(t), h(t)) \right)' f(t)\, dt \tag{7}$$

holds, and for $r, s > 1$ *the inequality is reversed.*

b) If $f : [a, b] \to \mathbf{R}$ is a nonnegative nonincreasing function then for $r < 1 < s$ (7) holds and for $r > 1 > s$ the inequality is reversed.

Proof: Let us suppose that $r, s < 1$ and f is nondecreasing. Using inequality (6) we obtain

$$\begin{aligned}
&M_p^{[r]}\left(\int_a^b g'(t)f(t)\,dt, \int_a^b h'(t)f(t)\,dt\right)\\
&\quad \le M_p^{[1]}\left(\int_a^b g'(t)f(t)\,dt, \int_a^b h'(t)f(t)\,dt\right)\\
&\quad = \int_a^b (p_1 g'(t) + p_2 h'(t))f(t)\,dt\\
&\quad = f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a))\\
&\qquad - \int_a^b M_p^{[1]}(g(t), h(t))\,df(t)\\
&\quad \le f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a))\\
&\qquad - \int_a^b M_p^{[s]}(g(t), h(t))\,df(t)\\
&\quad = f(b)M_p^{[1]}(g(b), h(b)) - f(a)M_p^{[1]}(g(a), h(a))\\
&\qquad - \Bigg(f(b)M_p^{[s]}(g(b), h(b)) - f(a)M_p^{[s]}(g(a), h(a))\\
&\qquad - \int_a^b (M_p^{[s]}(g(t), h(t)))' f(t)\,dt \Bigg)\\
&\quad = f(b)\left(M_p^{[1]}(g(b), h(b)) - M_p^{[s]}(g(b), h(b))\right)\\
&\qquad - f(a)\left(M_p^{[1]}(g(a), h(a)) - M_p^{[s]}(g(a), h(a))\right)\\
&\qquad + \int_a^p \left(M_p^{[s]}(g(t), h(t))\right)' f(t)\,dt\\
&\quad = \int_a^b \left(M_p^{[s]}(g(t), h(t))\right)' f(t)\,dt.
\end{aligned}$$

A similar proof applies in each of the other cases. □

Remark 1. In Theorem 1 we deal with two functions g and h. Obviously a similar result holds for n functions $x_1, \ldots, x_n$ which satisfy the same conditions as g and h.

Remark 2. It is obvious that on substituting $r = s = 0$ into (7) we have inequality (5) for $n = 2$. The result for $r = s = 0$ is given in [1].

In the following theorem we consider an inequality involving higher derivatives.

Theorem 2. *Let $f : [a, b] \to \mathbf{R}, x_i : [a, b] \to \mathbf{R} (i = 1, \ldots, m)$ be nonnegative functions with continuous n-th derivatives such that $x_i^{(n)}, (i = 1, \ldots, m)$ are nonnegative functions and $p_i, (i = 1, \ldots, m)$ be positive real numbers such that $\sum_{i=1}^{m} p_i = 1$.*

a) If $(-1)^{n-1} f^{(n)}$ is a nonnegative function, then for $r, s < 1$

$$\begin{aligned} &M_p^{[r]}\left(\int_a^b x_1^{(n)}(t) f(t)\,dt, \ldots, \int_a^b x_m^{(n)}(t) f(t)\,dt\right) \\ &\leq \Delta + \int_a^b \left(M_p^{[s]}(x_1(t), \ldots, x_m(t))\right)^{(n)} f(t)\,dt \end{aligned} \tag{8}$$

holds, where

$$\begin{aligned} \Delta = &\sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \\ &\left(\sum_{i=1}^{m} p_i x_i^{(k)}(t) - \left(M_p^{[s]}(x_1(t), \ldots, x_m(t))\right)^{(k)}\right)\Bigg|_a^b. \end{aligned}$$

If

$$x_i^{(k)}(a) = x_j^{(k)}(a) \text{ and } x_i^{(k)}(b) = x_j^{(k)}(b) \text{ for } i, j \in \{1, \ldots, m\} \tag{9}$$

and $k = 0, \ldots, n-1$, then

$$\begin{aligned} &M_p^{[r]}\left(\int_a^b x_1^{(n)}(t) f(t)\,dt, \ldots, \int_a^b x_m^{(n)}(t) f(t)\,dt\right) \\ &\leq \int_a^b \left(M_p^{[s]}(x_1(t), \ldots x_m(t))\right)^{(n)} f(t)\,dt. \end{aligned} \tag{10}$$

If $r, s > 1$, then the inequalities (8) and (10) are reversed.

b) If $(-1)^n f^{(n)}$ is a nonnegative function, then for $r < 1 < s$ the inequalities (8) and (10) hold and for $r > 1 > s$ they are reversed.

Proof: a) Let r and s be less than 1. Integrating by part n-times and using (6), we obtain

$$
\begin{aligned}
&M_p^{[r]}\left(\int_a^b x_1^{(n)}(t)f(t)\,dt,\ldots,\int_a^b x_m^{(n)}(t)f(t)\,dt\right)\\
&\le M_p^{[1]}\left(\int_a^b x_1^{(n)}(t)f(t)\,dt,\ldots,\int_a^b x_m^{(n)}(t)f(t)\,dt\right)\\
&= \left(\sum_{k=0}^{n-1}(-1)^{n-k1} f^{(n-k-1)}(t)\sum_{i=1}^m p_i x_i^{(k)}(t)\right)\Bigg|_a^b\\
&\quad -\int_a^b M_p^{[1]}(x_1(t),\ldots,x_m(t))(-1)^{(n-1)} f^{(n)}(t)\,dt\\
&\le \left(\sum_{k=0}^{n-1}(-1)^{n-k1} f^{(n-k-1)}(t)\sum_{i=1}^m p_i x_i^{(k)}(t)\right)\Bigg|_a^b\\
&\quad -\int_a^b M_p^{[s]}(x_1(t),\ldots,x_m(t))(-1)^{(n-1)} f^{(n)}(t)\,dt\\
&= \Delta + \int_a^b \left(M_p^{[s]}(x_1(t),\ldots,x_m(t))\right)^{(n)} f(t)\,dt.
\end{aligned}
$$

We shall prove that $\Delta = 0$ if $x_i, i = 1,\ldots,m$, satisfy (9).

Let us use notation $A_k = x_i^{(k)}(a)$ for $k = 0,1,\ldots,n-1$. Then $\sum_{i=1}^m p_i x_i^{(k)}(a) = A_k$. Consider the k-th order derivative of function y^p where y is an arbitrary function with k-th order derivative. First, there exists function $\phi_k^{[p]}$ such that

$$(y^p)^{(k)} = \phi_k^{[p]}(y, y', \ldots, y^{(k)}).$$

This follows by induction on k. For $k = 1$ we have $(y^p)' = py^{p-1}y' = \phi_1^{[p]}(y, y')$. Suppose that proposition is valid for all $j < k+1$. Then using Leibniz's rule we get

$$
\begin{aligned}
(y^p)^{(k+1)} &= (py^{p-1}\cdot y')^{(k)}\\
&= p\sum_{j=0}^{k}\binom{k}{j}(y^{p-1})^{(j)}(y')^{(k-j)}\\
&= p\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[p-1]}(y, y', \ldots, y^{(j)})\,y^{(k-j+1)}\\
&= \phi_{k+1}^{[p]}(y, y', \ldots, y^{(k+1)}).
\end{aligned}
\tag{11}
$$

Suppose that $s \neq 0$ and use the abbreviated notation $M(t)$ for the mean $M_p^{[s]}(x_1(t), \dots, x_m(t))$. Then $M^s(t) = \sum_{i=1}^m P_i x_i^s(t)$. The statement "$M^{(k)}(a) = A_k$" will be proved by induction on k. It is easy to check for $k = 0$ and $k = 1$.

Suppose it holds for all $j < k + 1$. Then

$$\left.\left(\sum_{i=1}^m p_i x_i^s(t)\right)^{(k+1)}\right|_{t=a} = \left.\sum_{i=1}^m p_i \phi_{(k+1)}^{[s]}\left(x_i(t), x_i'(t), \dots, x_i^{(k+1)}(t)\right)\right|_{t=a}$$
$$= \phi_{(k+1)}^{[s]}(A_0, A_1, \dots, A_{k+1})$$
$$= s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(A_0, A_1, \dots, A_j) A_{k-j+1}$$
$$+ \phi_k^{[s-1]}(A_0, A_1, \dots, A_k) A_{k+1}.$$

On the other hand, using (11) we get

$$(M^s(t))^{(k+1)}|_{t=a} = s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(M(a), M'(a), \dots, M^{(j)}(a))$$
$$\times M^{(k-j+1)}(a) + \phi_k^{[s-1]}(M(a), M'(a), \dots, M^{(k)}(a)) M^{(k+1)}(a)$$
$$= s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(A_0, A_1, \dots, A_j) A_{k-j+1} + \phi_k^{[s-1]}$$
$$(A_0, A_1, \dots, A_k) M^{(k+1)}(a).$$

Comparing these two results we obtain that $M^{(k+1)}(a) = A_{k+1}$, which is enough to conclude that $\Delta = 0$.

In the other cases the proof is similar, except in the case $s = 0$ which is left to the reader. $\square$

3. Applications

Now we will restrict our attention to the case when $r = 0$ and the x_i are power functions.

The case when $n = 1$.
Set: $r = 0, n = 1, a = 0, b = 1, x_i(t) = t^{a_i p_i + 1}$ in (8), where $a_i > -\frac{1}{p_i}$ for $i = 1, \dots, m, p_i > 0$ and $\sum_{i=1}^m \frac{1}{p_i} = 1$. We obtain that $\Delta = 0$ and

$$\int_0^1 t^{a_1 + \cdots + a_m} f(t)\, dt \geq \frac{\Pi_{i=1}^m (a_i p_i + 1)^{1/p_i}}{1 + \Sigma_{i=1}^m a_i} \prod_{i=1}^m \left(\int_0^1 t^{a_i p_i} f(t)\, dt \right)^{1/p_i}, \tag{12}$$

if f is a nondecreasing function. It is an improvement of Pólya's inequality (4). Some other results related to this inequality can be found in [5] and [8].

For example, combining (12) and the inequality

$$\sum_{i=1}^{m} a_i + 2 \geq \prod_{i=1}^{m} (a_i p_i + 2)^{1/p_i}$$

which follows from the inequality between arithmetic and geometric means, we obtain

$$\int_0^1 t^{a_1+\cdots+a_m} f(t)\, dt \geq \frac{\Pi_{i=1}^{m}((a_i p_i + 1)(a_i p_i + 2))^{1/p_i}}{(1+\Sigma_{i=1}^{m} a_i)(2+\Sigma_{i=1}^{m} a_i)} \times \prod_{i=1}^{m} \left(\int_0^1 t^{a_i p_i} f(t)\, dt \right)^{1/p_i}. \tag{13}$$

The case when $n = 2$.

Set: $r = 0, n = 2, a = 0, b = 1, x_i(t) = t^{a_i p_i + 2}$ in (8), where $a_i > -\frac{1}{p_i}$ for $i = 1, \ldots, m, p_i > 0$ and $\Sigma_{i=1}^{m} \frac{1}{p_i} = 1$. After some simple calculation, we obtain that $\Delta = 0$ and inequality (13) holds if f is a concave function. So inequality (13) applies not only for f nondecreasing, but also for f concave.

4. Results for Quasiarithmetic Means

Definition 2. Let f be a monotone real function with inverse $f^{-1}, p = (p_1, \ldots, p_n) = (p_i)_i, a = (a_1, \ldots, a_n) = (a_i)_i$ be real n-tuples. The quasiarithmetic mean of n-tuple a is defined by

$$M_f(a; p) = f^{-1}\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i f(a_i) \right),$$

where $P_n = \sum_{i=1}^{n} p_i$.

For $p_i \geq 0, P_n = 1, f(x) = x^r (r \neq 0)$ and $f(x) = \ln x (r = 0)$ the quasiarithmetic mean $M_f(a; p)$ is the weighted mean $M_p^{[r]}(a)$ of order r.

Theorem 3. *Let p be a positive n-tuple,* $x_i : [a, b] \to \mathbf{R}(i = 1, \ldots, n)$ *be nonnegative functions with continuous first derivative such that* $x_i(a) = x_j(a), x_i(b) = x_j(b), i, j = 1, \ldots, n$

a) If φ is a nonnegative nondecreasing function on $[a, b]$ and if f and g are convex increasing or concave decreasing functions, then

$$M_f\left(\left(\int_a^b x_i'(t)\varphi(t)\, dt \right)_i ; p \right) \geq \int_a^b M_g'((x_i(t))_i; p)\varphi(t)\, dt. \tag{14}$$

If f and g are concave increasing or convex decreasing functions, the inequality is reversed.

b) If φ is a nonnegative nonincreasing function on $[a, b]$, f convex increasing or concave decreasing function and g is concave increasing or convex decreasing, then (14) holds.

If f is concave increasing or convex decreasing function and g is convex increasing or concave decreasing, then (14) is reversed.

Proof: Suppose that φ is nondecreasing and f and g are convex functions. We shall use integration by parts and the well-known Jensen inequality for convex functions. The latter states that if (p_i) is a positive n-tuple and $a_i \in I$, then for every convex function $f : I \to R$ we have

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i). \tag{15}$$

We have

$$\begin{aligned}
M_f\left(\left(\int_a^b x_i'(t)\varphi(t)\,dt\right)_i; p\right) &= f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(\int_a^b x_i(t)\varphi(t)\,dt\right)\right) \\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i \int_a^b x_i'(t)\varphi(t)\,dt = \int_a^b \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_i'(t)\right)\varphi(t)\,dt \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_i(t)\right) d\varphi(t) \\
&\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^{n} p_i g(x_i(t)\right)\right) d\varphi(t) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b M_g(x_i(t))_i; p)\,d\varphi(t) \\
&= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i(t)\varphi(t)\Big|_a^b - M_g((x_i(t))_i; p)\varphi(t)\Big|_a^b \\
&\quad + \int_a^b M_g'((x_i(t))_i; p)\varphi(t)\,dt = \int_a^b M_g'((x_i(t))_i; p)\varphi(t)\,dt. \qquad \square
\end{aligned}$$

Theorem 4. *Let $x_i, i = 1, \dots, n$, satisfy assumptions of Theorem 4 and let p be a real n-tuple such that*

$$p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \dots, n), \quad P_n > 0. \tag{16}$$

a) If φ is a nonnegative nonincreasing function on $[a, b]$ and if f and g are concave increasing or convex decreasing functions, then (14) holds, while if f and g are convex increasing or concave decreasing (14) is reversed.

b) If φ is a nonnegative nondecreasing function on $[a,b]$, f is convex increasing or concave decreasing and g concave increasing or convex decreasing, then (14) holds.

If f is concave increasing or convex decreasing and g is convex increasing or concave decreasing, then (14) is reversed.

The proof is similar to that of Theorem 4. Instead of Jensen's inequality, a reverse Jensen's inequality [3, p. 6] is used: that is, if p_i is real n-tuple such that (16) holds, $a_i \in I, i = 1, \ldots, n$, and $(1/P_n)\sum_{i=1}^{n} p_i a_i \in I$, then for every convex function $f : I \to R$ (15) is reversed.

Remark 3. In Theorem 4 and 5 we deal with first derivatives. We can state an analogous result for higher-order derivatives as in Section 2.

Remark 4. The assumption that p is a positive n-tuple in Theorem 4 can be weakened to p being a real n-tuple such that

$$0 \le \sum_{i=1}^{k} p_i \le P_n \quad (1 \le k \le n), \quad P_n > 0$$

and $\left(\int x_i'(t)\varphi(t)\,dt\right)_i$ and $\left(x_i(t)\right)_i$, $t \in [a,b]$ being monotone n-tuples.

In that case, we use Jensen-Steffenen's inequality [3, p. 6]. instead of Jensen's in-equality in the proof.

In Theorem 5, the assumption on n-tuple p can be replaced by p being a real n-tuple such that for some $k \in \{1, \ldots, m\}$

$$\sum_{i=1}^{k} p_i \le 0 (k < m) \quad \text{and} \quad \sum_{i=k}^{n} p_i \le 0 (k > m)$$

and $\left(\int x_i'(t)\varphi(t)\,dt\right)_i$, $\left(x_i(t)\right)_i$, $t \in [a,b]$ being monotone n-tuples.

We use the reverse Jensen-Steffensen's inequality (see [3, p. 6] and [4]) in the proof.

5. Results for Logarithmic Means

We define the logarithmic mean $L_r(x,y)$ of distinct positive numbers x, y by

$$L_r(x,y) = \begin{cases} \left(\dfrac{1}{y-x}\dfrac{y^{r+1}-x^{r+1}}{r+1}\right)^{1/r} & r \ne -1, 0 \\ \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{\frac{1}{y-x}} & r = 0 \\ \dfrac{\ln y - \ln x}{y-x} & r = -1 \end{cases}$$

and take $L_r(x,x) = x$. The function $r \mapsto L_r(x,y)$ is nondecreasing.

It is easy to see that $L_1(x,y) = \frac{x+y}{2}$ and using method similar to that of the previous theorems we obtain the following result.

Theorem 5. *Let $g, h : [a,b] \mapsto \mathbf{R}$ be nonnegative nondecreasing functions with continuous first derivatives and $g(a) = h(a), g(b) = h(b)$.*

a) If f is a nonnegative increasing function on $[a,b]$, and if $r, s \le 1$, then

$$L_r\left(\int_a^b g'(t)f(t)\,dt, \int_a^b h'(t)f(t)\,dt\right) \le \int_a^b L_s'(g(t), h(t)f(t)\,dt. \quad (16)$$

If $r, s \ge 1$ then the reverse inequality holds.

b) If f is a nonnegative nonincreasing function then for $r < 1 < s$ (16) holds, and for $r > 1 > s$ the reverse inequality holds.

Proof: Let f be a nonincreasing function and $r < 1 < s$. Using $F = -f$, integration by parts and inequalities between logarithmic means we get

$$\begin{aligned}
&L_r\left(\int_a^b g'(t)f(t)\,dt, \int_a^b h'(t)f(t)\,dt\right)\\
&\le L_1\left(\int_a^b g'(t)f(t)\,dt, \int_a^b h'(t)f(t)\,dt\right) = \frac{1}{2}\int_a^b (g(t)+h(t))'f(t)\,dt\\
&= \frac{1}{2}(g(t)+h(t))f(t)\big|_a^b + \int_a^b \frac{1}{2}(g(t)+h(t))\,dF(t)\\
&\le \frac{1}{2}(g(t)+h(t))f(t)\big|_a^b + \int_a^b L_s(g(t),h(t))\,dF(t)\\
&= \frac{1}{2}(g(t)+h(t))f(t)\big|_a^b - L_s(g(t),h(t))f(t)\big|_a^b\\
&\quad + \int_a^b L_s'(g(t),h(t))f(t)\,dt = \int_a^b L_s'(g(t),h(t))f(t)dt.
\end{aligned}$$

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