

On the Congruences of the Tangents to a Surface

By

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Abstract

In this paper, for the congruences generated by the tangents to the lines of main curvature on a surface, necessary and sufficient conditions that the straight lines of these congruences are preserving the lines of curvature on their focal surfaces are derived and explained. Finally, two rectilinear congruences are introduced and investigated.

1. Introduction

The ambient space is the Euclidean space E^3 and for our work we have used [1,3,6] as general references. Let the vector function $\mathbf{r} = \mathbf{r}(u, v)$ represent a regular surface Φ and suppose that the parameters u, v are curvature parameters, i.e. the elements g_{12} and h_{12} of the first and second fundamental forms vanish identically ($g_{12} = h_{12} = 0$).

A line congruence in the Euclidean space E^3 is represented by the vector equation

$$\mathbf{R}(u, v, t) = \mathbf{r}(u, v) + t\mathbf{d}(u, v), t \in \mathfrak{R} \quad (1.1)$$

where $\mathbf{r}(u, v)$ is its base surface (the surface of reference) and $\mathbf{d}(u, v)$ is the unit vector giving the direction of the straight lines of the congruence.

Consider now the unit vector $\mathbf{e}_1(u, v)$, $\mathbf{e}_2(u, v)$ of the tangents of the parametric curves $v = \text{const.}$, $u = \text{const.}$ and the unit vector $\mathbf{e}_3(u, v)$ of the normal to the surface Φ at any arbitrary regular point, then we have

$$\mathbf{e}_1 = \frac{\mathbf{r}_u}{\sqrt{g_{11}}}, \mathbf{e}_2 = \frac{\mathbf{r}_v}{\sqrt{g_{22}}}, \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \quad (1.2)$$

which are invariant functions on the surface and \times is the usual vectorial product in E^3 .

Since the parameter net consists of lines of curvature on this surface we can calculate $ds = \sqrt{g_{11}} du$, $d\bar{s} = \sqrt{g_{22}} dv$, the arc-length parameters of the curves $v = \text{const.}$, $u = \text{const.}$, respectively. The moving frame $\{\mathbf{e}_i = \mathbf{e}_i(u, v), i = 1, 2, 3\}$ on the surface Φ at every regular point is then called DARBOUX frame. Hence, by means of the derivatives with respect to the arc-length parameter of the curves $v = \text{const.}$ with tangent \mathbf{e}_1 on the surface, the derivative formula with respect to the DARBOUX frame, may be stated as:

$$\frac{\partial}{\partial S} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & q & k \\ -q & 0 & 0 \\ -k & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \quad (1.3)$$

where $k = \frac{b_{11}}{g_{11}}$, $q = \frac{-(g_{11})_v}{2g_{11}\sqrt{g_{22}}}$ are the normal and geodesic curvatures of the curves $v = \text{const.}$ respectively. Similarly, the derivative formula of the DARBOUX frame of the curves $u = \text{const.}$ with tangent \mathbf{e}_2 on the surface is

$$\frac{\partial}{\partial \bar{S}} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{q} & 0 \\ -\bar{q} & 0 & \bar{k} \\ 0 & -\bar{k} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \quad (1.4)$$

also, $\bar{k} = \frac{b_{22}}{g_{22}}$, $\bar{q} = \frac{(g_{22})_u}{2g_{22}\sqrt{g_{11}}}$ have the same meaning as in (1.3) for the curves $u = \text{const.}$ on the surface. Here, g_{jk} and b_{jk} are the coefficients of the first and second fundamental forms of the surface Φ .

Since, k, \bar{k}, q, \bar{q} are invariant quantities of the curvature lines on the surface Φ , then these invariants and their derivatives must fulfil the Gauss-Codazzi equations, [1].

2. Main Results

It is known that on every regular surface there exists an orthogonal net such that the tangents to the surface along one family of this net form a

congruence, having this surface as one of its focal surfaces. Thus, we have the congruences:

$$T_1 : \mathbf{Y}(u, v, t) = \mathbf{r}(u, v) + t\mathbf{e}_1(u, v), \quad (2.1)$$

$$T_2 : \bar{\mathbf{Y}}(u, v, t) = \mathbf{r}(u, v) + t\mathbf{e}_2(u, v), \quad (2.2)$$

in view of (1.1). Such congruences are generated by these tangents of the surface Φ which belong to the DARBOUX frame at any regular point.

At first, we will investigate the congruence T_1 . The equation of the focal surfaces are obtained by equation (2.1) if we substitute in it successively the roots of the quadratic equation

$$[\mathbf{Y}_u, \mathbf{Y}_v, \mathbf{Y}_t] = 0, \quad (2.3)$$

where $\mathbf{Y}_u = \mathbf{r}_u + t\mathbf{e}_{1u}$, $\mathbf{Y}_v = \mathbf{r}_v + t\mathbf{e}_{1v}$, $\mathbf{Y}_t = \mathbf{e}_1$ [2]. In virtue of (1.3), (1.4), the equation of the focal surfaces, $\Phi_1 \equiv \Phi$, Φ_2 are:

$$\Phi : \mathbf{r} = \mathbf{r}(u, v), \Phi_2 : \mathbf{z}(u, v) = \mathbf{r}(u, v) - \frac{1}{\bar{q}}\mathbf{e}_1(u, v), (\bar{q} \neq 0). \quad (2.4)$$

From this we obtain that the second focal surface Φ_2 is really different from Φ .

Now we prove that the torsal surfaces of T_1 touch the focal surfaces along lines of curvature. For this purpose here, first we need the following derivatives:

$$\left. \begin{aligned} \mathbf{z}_1 &= \frac{\partial \mathbf{z}}{\partial \bar{S}} = \frac{(\bar{q}^2 + \bar{q}_1)}{\bar{q}^2} \mathbf{e}_1 - \frac{1}{\bar{q}}(q\mathbf{e}_2 + k\mathbf{e}_3), \\ \mathbf{z}_2 &= \frac{\partial \mathbf{z}}{\partial \bar{S}} = \frac{\bar{q}_2}{\bar{q}^2} \mathbf{e}_1, \end{aligned} \right\} \quad (2.5)$$

by using (1.3), (1.4). Hence the unit normal vector of the focal surface Φ_2 is

$$\mathbf{h} = \frac{\mathbf{z}_1 \times \mathbf{z}_2}{\|\mathbf{z}_1 \times \mathbf{z}_2\|} = \frac{-k\mathbf{e}_2 + q\mathbf{e}_3}{\sqrt{k^2 + q^2}}. \quad (2.6)$$

From (2.6) we see that \mathbf{h} is the instantaneous screw axis (I.S.A.) of Darboux's frame along the curves $v = \text{const.}$ on the surface Φ .

Hence the following theorem can be formulated:

Theorem (2.1). During the motion of the DARBOUX frame along a line of curvature on a regular surface the I.S.A. is intersected orthogonally by the instantaneous tangent to this line of curvature. The position vector of the intersection point on this tangent is the center of

geodesic curvature of that line of curvature which is orthogonal to the path of the DARBOUX frame.

Also, in view of (1.3), (1.4), form (2.6) we have that:

$$\left. \begin{aligned} \mathbf{h}_1 &= \frac{\partial \mathbf{h}}{\partial S} = \left(\frac{q_1 \kappa - q \kappa_1}{\kappa^2 + q^2} \right) \left(\frac{q \mathbf{e}_2 + \kappa \mathbf{e}_3}{\sqrt{\kappa^2 + q^2}} \right), \\ \mathbf{h}_2 &= \frac{\partial \mathbf{h}}{\partial S} = \frac{\kappa \bar{q}}{\sqrt{\kappa^2 + q^2}} \mathbf{e}_1 + \kappa \left(\frac{\bar{q}_1 + \bar{q}^2}{\kappa^2 + q^2} \right) \left(\frac{q \mathbf{e}_2 + \kappa \mathbf{e}_3}{\sqrt{\kappa^2 + q^2}} \right). \end{aligned} \right\} \quad (2.7)$$

It is known that the necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normal along this curve forms a developable surface. Therefore, respectively, we immediately derive from (2.5) and (2.7) that

$$\kappa(q_1 \kappa - q \kappa_1)(\bar{q}^2 + \bar{q}_1) = 0, \quad \kappa \bar{q}_2(\bar{q}_1 + \bar{q}^2) = 0 \quad (2.8)$$

are the necessary and sufficient conditions that the curves $v = \text{const.}$, $u = \text{const.}$ are lines of curvature on the surface $\bar{\Phi}_2$.

For the congruence T_2 , according to KUMMER, the focal surfaces are:

$$\bar{\Phi}_1 \equiv \bar{\Phi}, \quad \bar{\Phi}_2 : \bar{\mathbf{r}}(u, v) = \mathbf{r}(u, v) + \frac{1}{q} \mathbf{e}_2(u, v), \quad (2.9)$$

where $q \neq 0$ and the unit normal of the second focal surface is

$$\bar{\mathbf{h}} = \frac{\bar{\kappa} \mathbf{e}_1 + \bar{q} \mathbf{e}_3}{\sqrt{\bar{\kappa}^2 + \bar{q}^2}}. \quad (2.10)$$

In the same manner, respectively, we get:

$$\bar{\kappa}(\bar{q}_2 \bar{\kappa} - \bar{q} \bar{\kappa}_2)(q^2 - q_2) = 0, \quad \bar{\kappa} q_1(q^2 - q_2) = 0, \quad (2.11)$$

as the necessary and sufficient conditions that the curves $u = \text{const.}$, $v = \text{const.}$ are lines of curvature on the surface $\bar{\Phi}_2$.

We will now investigate the conditions (2.8) and (2.11) in detail. For a surface on which the net is the orthogonal net of lines of curvature we have:

- (i) if $\kappa = 0, \bar{\kappa} = 0$, then the surface is a developable ruled surface,
- (ii) if $q = 0, \bar{q} = 0$, then the surface is called a moulding surface [3],
- (iii) if $\kappa_1 q - \kappa q_1 = 0, \bar{\kappa}_2 \bar{q} - \bar{\kappa} \bar{q}_2 = 0$, then the surface is formed from plane curves, and
- (iv) if $\kappa_1 = 0, \bar{\kappa}_2 \neq 0$ or $\bar{\kappa}_2 = 0, \kappa_1 \neq 0$, then the surface is called canal surface.

To make sure that the second focal surface of T_1 is non degenerate we have to exclude developable(ruled) surfaces and moulding surfaces as base surfaces Φ . This means that $q \neq 0$ and $\bar{q} \neq 0$. It is known that for the developable ruled surfaces one of the families of lines of curvature consists of the generators of the surface. If the DARBOUX frame translates along one generator, then the I.S.A. does not exist and consequently, \mathbf{h} and $\bar{\mathbf{h}}$ are not defined. Therefore we exclude developable surfaces as base surfaces Φ which means that we assume $q \neq 0, \bar{q} \neq 0$. Since the families of lines of curvature on a moulding surface are plane curves, Φ cannot be a moulding surface, that is $q \neq 0, \bar{q} \neq 0$ (In fact we imposed $q \neq 0, \bar{q} \neq 0$ in equations (2.4), (2.9)). On the other hand, since the families of lines of curvature will also be plane curves, then the surface Φ cannot possess these families, that is $\kappa_1 q - \kappa q_1 = 0, \bar{\kappa}_2 \bar{q} - \bar{\kappa} \bar{q}_2 \neq 0$. Finally, the family of lines of curvature on the canal surface are plane curves. Hence the following theorem is proved:

Theorem (2.2). For the congruences generated by the tangents to the lines of curvature on a surface, the necessary and sufficient conditions that their torsal parameter surfaces touch the (non degenerated) focal surfaces along lines of curvature are:

$$\bar{q}^2 + \bar{q}_1 = 0, \quad q^2 - q_2 = 0. \quad (q \neq 0, \bar{q} \neq 0) \quad (2.12)$$

2.1. Curvatures of the Focal Surfaces

If $g_{j\bar{k}}^i (i = 1, 2)$ are the elements of the first fundamental forms of the focal surfaces, defined by equations (2.4), (2.9) respectively, from (2.5) we obtain that

$$g_{11}^1 = \frac{g_{11}(q^2 + \bar{\kappa}^2)}{\bar{q}^2}, g_{12}^1 = 0, g_{22}^1 = \frac{g_{22}\bar{q}_2^2}{\bar{q}^4}, \quad (2.13)$$

and similarly from (2.9), we have that

$$g_{22}^2 = \frac{g_{22}q_1^2}{q^4}, g_{12}^2 = 0, \quad g_{11}^2 = \frac{g_{11}(\bar{q}^2 + \bar{\kappa}^2)}{q^2}. \quad (2.14)$$

Since the parameter curves on the focal surfaces are lines of curvature, we may write:

$$ds_i = \sqrt{g_{11}^i} du, \quad d\bar{s}_i = \sqrt{g_{22}^i} dv, \quad (i = 1, 2) \quad (2.15)$$

as the arc-length parameters of the curves $v = \text{const.}$ $u = \text{const.}$, respectively. If we consider now, $\mathbf{f}_1 = \frac{\mathbf{Z}_u}{\sqrt{g_{11}^i}}, \mathbf{f}_2 = \frac{\mathbf{Z}_v}{\sqrt{g_{22}^i}}$ the unit vectors of the

tangents of the parametric curves on the focal surface (2.4), then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{h}\}$ is the connected Darboux frame trihedron on this surface, where $\mathbf{f}_1 = -\frac{q\mathbf{e}_2 + k\mathbf{e}_3}{\sqrt{q^2 + k^2}}$, $\mathbf{f}_2 = \mathbf{e}_1$ and $\mathbf{h} = \mathbf{f}_1 \times \mathbf{f}_2$. According to the above, $\mathbf{f}_1, \mathbf{f}_2$ are called the principal directions on the focal surface (2.4). In fact, using the derivatives of \mathbf{h} are formed with respect to \bar{s}_1, s_1 , equations (2.7) get the form:

$$\left. \begin{aligned} \frac{\partial \mathbf{h}}{\partial s_1} &= \frac{\partial \mathbf{h}}{\partial s} \cdot \frac{\partial s}{\partial s_1} = -\frac{\bar{q}(q_1 k - q k_1)}{(k^2 + q^2)^{3/2}} \mathbf{f}_1, \\ \frac{\partial \mathbf{h}}{\partial \bar{s}_1} &= \frac{\partial \mathbf{h}}{\partial \bar{s}} \cdot \frac{\partial \bar{s}}{\partial \bar{s}_1} = \frac{k \bar{q}^3}{\bar{q}_2 \sqrt{k^2 + q^2}} \mathbf{f}_2. \end{aligned} \right\} \quad (2.16)$$

Hence we find, from (2.16), that the Gaussian and mean curvatures of the focal surface (2.4) are:

$$K = -\frac{k \bar{q}^4 (q_1 k - q k_1)}{\bar{q}_2 (k^2 + q^2)^2}, \quad (2.17)$$

and

$$H = \frac{\bar{q} [\bar{q}_2 (q k_1 - q_1 k) + k \bar{q}^2 (k^2 + q^2)]}{2 \bar{q}_2 (k^2 + q^2)^{3/2}}. \quad (2.18)$$

The same argument is valid for the surface defined by (2.9), hence the Gaussian and mean curvatures are:

$$\bar{K} = -\frac{k \bar{q}^2 (\bar{q}_2 \bar{k} - \bar{q} \bar{k}_2)}{q_1 (\bar{k}^2 + \bar{q}^2)^2}, \quad (2.17)'$$

and

$$\bar{H} = -\frac{q [q_1 (\bar{q} \bar{k}_2 - \bar{k} \bar{q}_2) + \bar{k} \bar{q}^2 (\bar{k}^2 + \bar{q}^2)]}{2 q_1 (\bar{k}^2 + \bar{q}^2)^{3/2}}. \quad (2.18)'$$

Hence, from (2.17) and (2.17)' the following theorem is proved.

Theorem (2.3). For the congruence generated by the tangents to the lines of curvature on a surface, their torsal parameter surfaces touch the (non degenerated) second focal surface along lines of curvature, the

base surface can neither be a canal surface, nor a moulding surface or a non developable surface. The second focal surface cannot be a developable or moulding or canal type either.

3. The Instantaneous Rectilinear Congruences

Let us start again with the regular surface Φ defined at the beginning of our work. It is known that the consecutive normals along a line of curvature intersect, the points of intersection being the corresponding center of curvature. The locus of the centers of curvature for all points of the surface is called the surface of centers of centra-surface of. It consists of two sheets, corresponding to the two families of lines of curvature. Therefore, the equations of those surfaces are:

$$\Psi : \mathbf{y}(u, v) = \mathbf{r}(u, v) + \frac{1}{k(u, v)} \mathbf{e}_3(u, v), \quad (3.1)$$

$$\bar{\Psi} : \bar{\mathbf{y}}(u, v) = \mathbf{r}(u, v) + \frac{1}{\bar{k}(u, v)} \mathbf{e}_3(u, v). \quad (3.2)$$

The problem we are going to investigate is the following: By the motion of Darboux's frame on the surface Φ there are two rectilinear congruences generated by the instantaneous screw axes $b(u, v), \bar{b}(u, v)$.

First we deal with vectors in the Euclidean space E^3 from the purely geometrical point of view. For this purpose, consider the line joining the points of the surfaces $\Psi, \bar{\Psi}_2$ (defined by equation (2.9)): if $\bar{\mathbf{z}}$ is designated as the initial point while the other endpoint \mathbf{y} as the terminal point, and we have the oriented segment

$$\bar{\mathbf{z}}\mathbf{y} = \mathbf{y} - \bar{\mathbf{z}}. \quad (3.3)$$

From (3.3), the unit vector $\frac{\bar{\mathbf{z}}\mathbf{y}}{\|\bar{\mathbf{z}}\mathbf{y}\|}$ along this segment, is the instantaneous screw axis $\mathbf{h}(u, v)$, defined by equation (2.6). The same can be done for the surfaces $\bar{\Psi}, \Phi_2$ (defined by equation (2.4)).

Therefore we can introduce the following rectilinear congruences:

$$I : \mathbf{Q}(u, v, t) = \mathbf{r}(u, v) + \frac{1}{q(u, v)} \mathbf{e}_2(u, v) + t\mathbf{h}(u, v), \quad (3.4)$$

$$\bar{I} : \bar{\mathbf{Q}}(u, v, t) = \mathbf{r}(u, v) - \frac{1}{\bar{q}(u, v)} \mathbf{e}_1(u, v) + t\bar{\mathbf{h}}(u, v), \quad (3.5)$$

3.1. Properties of the Congruence $\mathbf{I}, \bar{\mathbf{I}}$

If $e, f, g; a, b, b', c$ are the coefficients of the first and second fundamental forms of \mathbf{I} , in Kummer's sense, we have from equations (2.7), (2.9) that

$$\left. \begin{aligned}
 e &:= \langle \mathbf{h}_u, \mathbf{h}_u \rangle \\
 &= g_{11} \langle \mathbf{h}_1, \mathbf{h}_1 \rangle = g_{11} \left(\frac{q_1 k - q k_1}{k^2 + q^2} \right)^2, \\
 f &:= \langle \mathbf{h}_u, \mathbf{h}_v \rangle \\
 &= \sqrt{g_{11} g_{22}} \langle \mathbf{h}_1, \mathbf{h}_2 \rangle = \sqrt{g_{11} g_{22}} \frac{k(q_1 k - q k_1)(\bar{q}_1 + \bar{q}^2)}{(k^2 + q^2)^2}, \\
 g &:= \langle \mathbf{h}_v, \mathbf{h}_v \rangle \\
 &= g_{22} \langle \mathbf{h}_2, \mathbf{h}_2 \rangle = g_{22} \frac{k^2[\bar{q}^2(k^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2]}{(k^2 + q^2)^2},
 \end{aligned} \right\} \quad (3.6)$$

and

$$\left. \begin{aligned}
 a &:= \langle \mathbf{h}_u, \bar{\mathbf{z}}_u \rangle \\
 &= g_{11} \langle \mathbf{h}_1, \bar{\mathbf{z}}_1 \rangle = -g_{11} \frac{q_1(q_1 k - q k_1)}{q(k^2 + q^2)^{3/2}}, \\
 b &:= \langle \mathbf{h}_u, \bar{\mathbf{z}}_v \rangle \\
 &= \sqrt{g_{11} g_{22}} \langle \mathbf{h}_1, \bar{\mathbf{z}}_2 \rangle = \sqrt{g_{11} g_{22}} \frac{(q_1 k - q k_1)(\bar{q}_1 + \bar{q}^2)}{(k^2 + q^2)^{3/2}}, \\
 b' &:= \langle \mathbf{h}_v, \bar{\mathbf{z}}_u \rangle \\
 &= \sqrt{g_{11} g_{22}} \langle \mathbf{h}_2, \bar{\mathbf{z}}_1 \rangle = -\sqrt{g_{11} g_{22}} \frac{k q_1 (\bar{q}_1 + \bar{q}^2)}{q(k^2 + q^2)^{3/2}}, \\
 c &:= \langle \mathbf{h}_v, \bar{\mathbf{z}}_v \rangle \\
 &= g_{22} \langle \mathbf{h}_2, \bar{\mathbf{z}}_2 \rangle = -g_{22} \frac{k[(\bar{q}_1 + \bar{q}^2) + \bar{q}^2(k^2 + q^2)]}{q(k^2 + q^2)^{3/2}}.
 \end{aligned} \right\} \quad (3.7)$$

The same calculations can be valid for the rectilinear congruence $\bar{\mathbf{I}}$. Therefore, if $\bar{e}, \bar{f}, \bar{g}; \bar{a}, \bar{b}, \bar{b}', \bar{c}$ are the coefficients of the first and second

fundamental forms of $\bar{\mathbf{I}}$, we have

$$\left. \begin{aligned} \bar{e} &= g_{11} \frac{\bar{k}^2 [q^2 (\bar{k}^2 + \bar{q}^2) + (q^2 - q_2)^2]}{(\bar{k}^2 + \bar{q}^2)^2}, \\ \bar{f} &= \sqrt{g_{11} g_{22}} \frac{\bar{k} (\bar{q}_2 \bar{k} - \bar{q} \bar{k}_2) (q^2 - q_2)}{(\bar{k}^2 + \bar{q}^2)^2}, \\ \bar{g} &= g_{22} \left(\frac{\bar{q}_2 \bar{k} - \bar{q} \bar{k}_2}{\bar{k}^2 + \bar{q}^2} \right)^2, \end{aligned} \right\} (3.6)'$$

and

$$\left. \begin{aligned} \bar{a} &= g_{11} \bar{k} \frac{[(q^2 - q_2)^2 - q^2 (\bar{k}^2 + \bar{q}^2)]}{\bar{q} (\bar{k}^2 + \bar{q}^2)^{3/2}}, \\ \bar{b} &= \sqrt{g_{11} g_{22}} \frac{\bar{k} \bar{q}_2 (q^2 - q_2)}{\bar{q} (\bar{k}^2 + \bar{q}^2)^{3/2}}, \\ \bar{b}' &= \sqrt{g_{11} g_{22}} \frac{(\bar{q}_2 \bar{k} - \bar{q} \bar{k}_2) (q^2 - q_2)}{\bar{q} (\bar{k}^2 + \bar{q}^2)^{3/2}}, \\ \bar{c} &= g_{22} \frac{\bar{q}_2 (\bar{q}_2 \bar{k} - \bar{q} \bar{k}_2)}{\bar{q} (\bar{k}^2 + \bar{q}^2)^{3/2}}. \end{aligned} \right\} (3.7)'$$

It is clear now from equations (3.7), (3.7)' that $b = b' = 0$ and $\bar{b} = \bar{b}' = 0$, i.e. the rectilinear congruences \mathbf{I} , $\bar{\mathbf{I}}$ are of normal type if and only if condition (2.12) is satisfied.

Hence the following theorem is proved:

Theorem (3.1). Consider a rectilinear congruence generated by the tangents to a surface along one set of lines of curvature. If the generators of this congruence are preserving the lines of curvature on their focal surfaces, then the I.S.A. of the Darboux frame, along this set of lines of curvature, will generate a normal rectilinear congruence, i.e. $b = b' = 0$ [6].

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