Sitzungsber. Abt. II (1999) 208: 115–122

# Base Changes for (t, m, s)-Nets and Related Sequences

By

**G. Pirsic***

(Vorgelegt in der Sitzung der math.-nat. Klasse am 15. April 1999
durch das w. M. Harald Niederreiter)

**Abstract**

We consider finite and infinite point sets of low discrepancy, $(t, m, s)$- nets and $(\mathbf{T}, s)$-sequences. The parameters, $t$, $\mathbf{T}$ are directly related to the discrepancy of the point sets and thus indicative of their quality.

The definitions of these point sets fundamentally involve a further natural number $b > 1$ called the base. We examine single instances of the above objects with respect to different bases and try to reassess the quality parameters $t$, $\mathbf{T}$. Upper estimates of the new respective parameters $t'$, $\mathbf{T}'$ are obtained by employing and, in some cases, explicitly calculating, sums of certain remainders. The ensuing estimates generalize and slightly improve known results of this type.

## 1. Motivation

For some time now, $(t, m, s)$-nets and $(\mathbf{T}, s)$-sequences have been recognized as point sets (sequences) of very low discrepancy and as being especially useful in quasi-Monte Carlo integration. Again in this area, the integration of rapidly convergent Walsh series (see [4], [5], [3], [2]) yields very good results.

When performing integration of Walsh series with a $(t, m, s)$-net, where Walsh functions in a base $b$ are used, it may not be appropriate or possible to use the same base for the net, whereas we still want to retain the good error estimates we have if the bases are equal.

*Work supported by the OeNB, Project 6788.

One approach is to consider a function, which is by assumption a rapidly convergent Walsh series in base $b$, as a Walsh series to another base $c$ for which good nets are easily obtainable and investigate its quality of convergence. This was done quite extensively in [8].

The 'dual' approach would be to examine the net according to its quality with respect to a different base. (This is an example of what is known as a 'propagation rule of nets'.) Some trivial relationships are immediate and already mentioned in early papers on the subject. In [7], Lemma 9 the idea of base change for $(t, m, s)$-nets was given consideration. We were able to slightly improve this result and moreover give a more general form that connects the problem with the evaluation of sums of certain remainders.

As in [7], the results are also applied to low-discrepancy sequences, namely, $(\mathbf{T}, s)$- and $(t,s)$-sequences.

## 2. Conventions and Definitions

The definition of $(t, m, s)$*-nets is as follows:*

**Definition 1.** *Let* $b \geq 2$, $s \geq 1$, *and* $0 \leq t \leq m$ *be integers. Then a point set* $P = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$ *consisting of* $N = b^m$ *points of* $[0,1)^s$ *forms a* $(t, m, s)$*- net in base b if the number of n with* $0 \leq n \leq N-1$ *for which* $\mathbf{x}_n$ *is in the subinterval J of* $[0,1)^s$*, is* $b^t$ *for all*

$$J = \prod_{i=1}^{s} \left[ \frac{a_i}{b^{d_i}}, \frac{(a_i+1)}{b^{d_i}} \right)$$

*with integers* $d_i \geq 0$ *and* $0 \leq a_i < b^{d_i}$ *for* $1 \leq i \leq s$, *and with s-dimensional volume* $b^{t-m}$.

*Intervals of the form of J will be called elementary intervals in base b.*

Note that nets are finite point sets. An equivalent to $(t, m, s)$-nets in sequences are $(\mathbf{T}, s)$-sequences (first introduced in [1]):

**Definition 2.** *Let* $b \geq 2, s \geq 1$, *and* $\mathbf{T} : \mathbb{N} \mapsto \mathbb{N}_0$ *be a function such that* $\mathbf{T}(m) \leq m$. *A sequence* $\{x_i\}_{i=0}^{\infty}$ *shall be called a* $(\mathbf{T}, s)$*-sequence in base b if for every* $m \in \mathbb{N}, k \in \mathbb{N}_0$, *the point sets* $\{x_i\}_{i=kb^m}^{(k+1)b^m-1}$ *are* $(\mathbf{T}(m), m, s)$*-nets in base b.*

*If the function* $\mathbf{T}$ *is constant, say,* $\mathbf{T}(m) = t \in \mathbb{N}_0 (\forall m \in \mathbb{N})$, *we obtain an important special case, which is called a* $(t, s)$*-sequence in base b.*

We introduce variables consistently used in this paper:

**Convention 1.** *Throughout this paper we shall consider two bases which are powers of the same integer b and designate them as* $b^h$ *and* $b^{h'}$*, where b, h,* $h' \in \mathbb{N}, b \geq 2, h, h' \geq 1$. *Further we will consider an arbitrary* $(t, m, s)$*-net P in base* $b^h (t, m, s \in \mathbb{N}_0, 0 \leq t \leq m; m, s \geq 1)$ *and ask for the optimal* $t'$ *such that P is a* $(t', m', s)$*-net in base* $b^{h'}$, where $hm = h'm'$ *and* $t', m' \in \mathbb{N}_0, 0 \leq t' \leq m', m' \geq 1$.

*Since for $h'' := (h, h')$ we can always write the bases as $(b^{h''})^{h/h''}(b^{h''})^{h'/h''}$, we will w.l.o.g. assume that $(h, h') = 1$.*

The function $M(t'')$ that we are going to define next is of essential importance in estimating the optimal $t'$.

**Definition 3.** *For some $t'' \in \{0, \dots, m'\}$ consider all partitions $\{d_i'\}_{i=1}^s$ of $m' - t''$ in $s$ parts (i.e. $\sum d_i' = m' - t''$, $d_i' \geq 0$), where the $d_i'$ are in ascending order. Set $r_i := (-h'd_i') \bmod h$ ($r_i \in \{0, \dots, h-1\}$) and take the sum of all $r_i$. By $M(t'')$ we will denote the maximal sum of $r_i$, taken over all partitions of $m' - t''$ in $s$ parts*:

$$M(t'') := \max\left\{ \sum_{i=1}^{s} r_i \,\middle|\, 0 \leq d_1' \leq \cdots \leq d_s' \leq m' - t'', \right.$$
$$\left. \sum_{i=1}^{s} d_i' = m' - t'', r_i = (-h'd_i') \bmod h \right\}.$$

## 3. Results

The first result we arrive at is the following:

**Theorem 1.** *Let $b \geq 2, h, m, h', m', s \geq 1$ and $0 \leq t \leq m$ be integers with $hm = h'm'$. Then every $(t, m, s)$-net in base $b^h$ is a $(t', m', s)$-net in base $b^{h'}$, where*

$$t' = \min\{t'' | h't'' - M(t'') \geq ht\}.$$

Although $M(t'')$ and thus $t'$ are finitely computable, the calculatory effort is quite high since we have to compute all (ordered) partitions first. We give the following estimate for $M(t'')$.

**Lemma 1.** *For $b, h, m, h', m', s, t, M(t'')$ as in Theorem 1, we have*

$$M(t'') \leq \min\{(-h' \bmod h)(m' - t''), (s-1)(h-1) + (h't'' + s - 1 \bmod h)\}.$$

*For $h$=1, 2, 3, 4 and for $h' \equiv 1$ or $-1$ (mod $h$), the inequality is an equality. (Also for the weaker condition $h'^{-1} \bmod h \leq (s+1)/(s-1)$.)*

Applying the estimate to Theorem 1 leads to

**Corollary 1.** *For $b, h, m, h', m', s, t$ as in Theorem 1, every $(t, m, s)$-net in base $b^{h'}$ is a a $(t', m', s)$-net in base $b^{h'}$, where*

$$t' = \min\left\{ \left\lceil \frac{ht + (s-1)(h-1)}{h'} \right\rceil, \left\lceil \frac{ht + m'(-h' \bmod h)}{h' + (-h' \bmod h)} \right\rceil \right\} \leq m'$$

**Remark 1.** *The second term in the preceding corollary prevails over the first term whenever the dimension $s$ is high and $m$ is not too large. In more detail, the second term will be attained if the following inequality is fulfilled:*

$$ht + (h' + 1)(h-1)(s-1) \geq h^2 m$$

*(a short proof for this sufficient condition can be found in the Proofs section).*

*We illustrate this with a numerical example. If we take*

$$h = 3, h' = 2, t = 8, m = 12, s = 20,$$

*the inequality is fulfilled. Evaluation of the first term in the Corollary gives* 31, *but since the trivial bound* $m'$=18 *is already lower, this value must be rejected. The second term, however, leads to* $t'$=14 *which is* 4 *points below the trivial bound.*

**Remark 2.** *The first term in Corollary* 1 *can already be obtained from Lemma* 9 *in* [7] *(where* $h'$=1*) together with Lemma 2.9 in* [6] *(where* $h = 1$*). However, if we had applied Lemma* 1 *only for* $h'$=1 *and subsequently used Lemma* 2.9 *of* [6]*, the resulting second term*

$$\left\lceil \frac{t + m(h-1)}{h'} \right\rceil$$

*would be larger than the one given in Corollary* 1.

An application to $(\mathbf{T}, s)$- sequences can easily be given:

**Corollary 2.** *Any* $(\mathbf{T}, s)$*-sequence in base* $b^h$ *is a* $(\mathbf{T}', s)$*-sequence in base* $b^{h'}$*, where (with* $h'm' = hm + r$, $0 \le r < h$*):*

$$\mathbf{T}'(m') := \min\left\{ \left\lceil \frac{h\mathbf{T}(m) + (s-1)(h-1) + r}{h'} \right\rceil, \left\lceil \frac{\mathbf{T}(m) + m(h-1) + r}{h'} \right\rceil, m' \right\}.$$

**Remark 3.** *Obviously, a corresponding statement about* $(t, s)$*-sequences can be given by letting* $\mathbf{T}(m)$ *be a constant function and estimating the remainder r by the worst possible case* $h-1$. *Again, the combination of Propositions* 4 *and* 5 *in* [7] *already leads to the resulting first term* $\lceil (ht + (h-1)s)/h' \rceil$.

## 4. Proofs

*Proof of Theorem 1:* Given an arbitrary $(t, m, s)$-net $P$ in base $b^h$, we want to determine $t'$ such that $P$ is a $(t', m', s)$-net in base $b^{h'}$. To that end, we consider an elementary interval $J$ in base $b^{h'}$ of volume $(b^{h'})^{t''}/(b^{h'})^{m'}$, where $t'' \in \{0, \ldots, m'\}$ is arbitrary, and try to count the points inside $J$. Let

$$J = \prod_{i=1}^{s} \left[ \frac{a_i}{(b^{h'})^{d_i'}}, \frac{a_i + 1}{(b^{h'})^{d_i'}} \right),$$

where $a_i, d_i' \ge 0$ are integers and $\sum d_i' = m' - t''$.

Now, for all $i$, perform a division with remainder, defining $d_i, r_i : h'd_i' = hd_i - r_i, d_i, r_i \geq 0, r_i < h$. So for any $i$ we have:

$$\left[\frac{a_i}{(b^{h'})^{d_i'}}, \frac{a_i+1}{(b^{h'})^{d_i'}}\right) = \left[\frac{a_i}{b^{hd_i - r_i}}, \frac{a_i+1}{b^{hd_i-r_i}}\right) = \bigcup_{j=0}^{b^{r_i}-1} \left[\frac{a_i b^{r_i} + j}{(b^h)^{d_i}}, \frac{a_i b^{r_i} + j + 1}{(b^h)^{d_i}}\right).$$

Thus, by

$$J = \prod_{i=1}^{s} \bigcup_{j=0}^{b^{r_i}-1} \left[\frac{a_i b^{r_i} + j}{(b^h)^{d_i}}, \frac{a_i b^{r_i} + j + 1}{(b^h)^{d_i}}\right)$$
$$= \bigcup_{j_1=0}^{b^{r_1}-1} \cdots \bigcup_{j_s=0}^{b^{r_s}-1} \prod_{i=1}^{s} \left[\frac{a_i b^{r_i} + j_i}{(b^h)^{d_i}}, \frac{a_i b^{r_i} + j_i + 1}{(b^h)^{d_i}}\right),$$

$J$ is the union of $b^{\Sigma r_i}$ elementary intervals in base $b^h$, each of which has the volume

$$b^{-h\sum_{i=1}^{s} d_i} = b^{-(h'\sum_{i=1}^{s} d_i' + \sum_{i=1}^{s} r_i)} = b^{-(h'(m'-t'') + \sum_{i=1}^{s} r_i)}.$$

Now, by the definition of $(t, m, s)$-nets in base $b$, if any subset $S$ of $[0,1)^s$ is a union of nonoverlapping elementary intervals in base $b$, each of which has a volume at least $b^t/b^m$, the subset $S$ contains the "right" amount of points namely $b^m$ times the $s$-dimensional volume of $S$.

In our case, the volume $b^{-(h'(m'-t'')+\sum r_i)}$ of the single elementary components of $J$ has to be at least $(b^h)^t/(b^h)^m = b^{h(t-m)}$, that is,

$$-\left(h'(m' - t'') + \sum_{i=1}^{s} r_i\right) \geq h(t - m) \Leftrightarrow$$
$$h't'' - \sum_{i=1}^{s} r_i \geq ht.$$

So, if for any $t''$ this inequality holds for all elementary intervals $J$ (in base $b^{h'}$ with volume $b^{h'(m'-t'')}$), which is equivalent to saying that it holds for the maximal value of the sum over the $r_i$ (we defined $M(t'')$ to be just this maximal value), then the point set $P$ is a $(t'', m', s)$-net in the base $b^{h'}$. Finally, looking for the smallest such $t''$ gives us the claimed expression for $t'$:

$$t' = \min\{t'' | h't'' - M(t'') \geq ht\}.$$

□

*Proof of Lemma 1:* The first estimate of $M(t'')$ is arrived at easily:

$$M(t'') = \max\left\{ \sum_{i=1}^{s} r_i \,\middle|\, \sum_{i=1}^{s} d_i' = m' - t'', r_i = (-b'd_i')\text{mod } b \right\}$$
$$\leq \max\left\{ \sum_{i=1}^{s} (-b'\text{mod } b)d_i' \,\middle|\, \sum_{i=1}^{s} d_i' = m' - t'' \right\}$$
$$= (-b'\text{mod } b)(m' - t'').$$

For the second estimate observe that $M(t'') \leq s(b-1)$ and for any $r_1, \ldots, r_s$ as in the definition of $M(t'')$:

$$\sum_{i=1}^{s} r_i \equiv \sum_{i=1}^{s} -b'd_i' \equiv -b'(m' - t'') = b't'' - bm \equiv b't''(\text{mod } b),$$

which leads to the preliminary estimate

$$M(t'') \leq \max\{b't'' + kb | k \in \mathbb{Z}, b't'' + kb \leq s(b-1)\}.$$

Clearly, the possible range of values for the right hand side expression is from $(s-1)(b-1)$ to $s(b-1)$ since we can always add appropriate multiples of $b$ to get into this interval. Inside this interval we have one and only one value that is congruent to $b't''$ modulo $b$. We can put this specific value into analytical terms as is done in the right hand side of the next inequality, giving the second estimate.

$$M(t'') \leq (s-1)(b-1) + (b't'' - (s-1)(b-1)\text{mod } b)$$
$$= (s-1)(b-1) + (b't'' + s - 1\text{mod } b).$$

The first estimate is strict for all $t''$ such that $(m' - t'') \leq s$, since then we can choose $d_1' = d_2' = \cdots = d_{m'-t''}' = 1,\ d_{m'-t''+1}' = \cdots = d_s' = 0$, which makes the sum of the $r_i$ equal to $(m' - t'')(-b'\text{mod } b)$.

The second estimate is strict for all $t''$ such that

$$(m' - t'') \geq (s-1)(b'^{-1}\text{mod } b),$$

We can choose $d_1' = \cdots = d_{s-1}' = (b'^{-1}\text{mod } b)$ and

$$d_s' = (m' - t'') - (s-1)(b'^{-1}\text{mod } b),$$

which gives $r_1 = \cdots = r_{s-1} = (b-1)$ and $r_s = (b't'' + s - 1 \text{ mod } b)$, so that the sum of the $r_i$ evaluates to the second estimate.

As a consequence of the last two paragraphs the given estimate for $M(t'')$ is strict for $b, b'$ such that $b' \equiv 1$ (mod $b$). (We also have the slightly weaker sufficient condition $b'^{-1}$ mod $b \leq (s+1)/(s-1)$ for strictness of the estimate.)

For $b' \equiv -1 \pmod b$, note that $r_i = d_i' \bmod b$, so that for $(m'-t'') \le (s-1)(b-1) = (s-1)(b'^{-1} \bmod b)$ we can always reach the first estimate by choosing each $d_i'$ smaller than $b$, for then the sum of the $r_i$ equals the sum of the $d_i'$, which is $(m'-t'') = (m'-t'')(-b' \bmod b)$.

The last two paragraphs cover all cases of $b = 1,2,3,4$, since for $b = 1$ the function $M$ is trivially constant and equal to 0 and for $b > 1$ all relatively prime $b'$ are either congruent to 1 or $-1$ modulo $b$. □

*Proof of Corollary 1:* All we have to do is to plug in Lemma 1 into Theorem 1. For the first estimate of the Lemma (the second of the Corollary) we have:

$$b't'' - M(t'') \ge b't'' - (-b' \bmod b)(m' - t'') \ge bt \Leftrightarrow$$

$$t'' \ge \frac{bt + m'(-b' \bmod b)}{b' + (-b' \bmod b)},$$

$$t' = \min\{t'' | b't'' - M(t'') \ge bt\} \le \left\lceil \frac{bt + m'(-b' \bmod b)}{b' + (-b' \bmod b)} \right\rceil.$$

We also get $t' \le m'$ very quickly from the last line, observing $bt \le bm = b'm'$ and the monotonicity of the ceiling function.

As regards the other estimate, we have to find the smallest $t''$ that fulfills the right hand side inequality in

$$b't'' - M(t'') \ge b't'' - ((s-1)(b-1) + (b't'' + s - 1 \bmod b)) \ge bt.$$

We set $t'' = (bt + (s-1)(b-1) + F)/b'$, $F \in \mathbb{Z}$. The inequality then becomes

$$bt + (s-1)(b-1) + F - (s-1)(b-1) -$$
$$(bt + (s-1)(b-1) + F + s - 1 \bmod b) = bt + F - (F \bmod b) \ge bt.$$

Clearly, this inequality is fulfilled for all $F \ge 0$ and no $F < 0$. So

$$t' = \min\{t'' | b't'' - M(t'') \ge bt\} \le \left\lceil \frac{bt + (s-1)(b-1)}{b'} \right\rceil,$$

completing the proof. □

*Proof of Remark 1:* We want to establish a sufficient condition for the first term in the Corollary to be at least as large as the second one.

Initially we remove the ceiling brackets, using the monotonicity of the ceiling function:

$$\frac{bt + (s-1)(b-1)}{b'} \ge \frac{bt + m'(-b' \bmod b)}{b' + (-b' \bmod b)}.$$

Now we multiply with the denominators and cancel equal terms:

$$(-b' \bmod b)bt + (s-1)(b-1)(b' + (-b' \bmod b)) \geq bm(-b' \bmod b).$$

Finally we strengthen the inequality by using $(-b' \bmod b) \geq 1$ on the left hand side and $(-b' \bmod b) < b$ on the right hand side:

$$bt + (s-1)(b-1)(b'+1) \geq b^2 m. \qquad \square$$

*Proof of Corollary:* The given terms are obtained by first considering the base change where $b' = 1$, which can be done along the same lines as Proposition 4 in [7] (only now using our Corollary 1 instead of Lemma 9 in [7]) and afterwards applying Proposition 5 in [7], corresponding to $b = 1$. These propositions, stated for $(t, s)$-sequences, are obviously valid for $(\mathbf{T}, s)$-sequences as well if we replace $t$ by $\mathbf{T}(m)$ and do not estimate the remainder $r$ by the worst case $b - 1$. $\square$

## Acknowledgements

We would like to express our gratitude to the referee who pointed out several errors in the original submission. With his help we were able to significantly improve the quality, readability and correctness of this paper.

## References

[1] Larcher, G., Niederreiter, H.: Generalized $(t, s)$-sequence, Kronecker type sequences, and diophantine approximations of formal Laurent series. Trans. Am. Math. Soc. **347**, 2051–2073 (1995).

[2] Larcher, G., Pirsic, G.: Base change problems for generalized Walsh series and multivariate numerical integration. Pacific J. Math. 189–191 (1999).

[3] Larcher, G., Pirsic, G., Wolf, R.: Quasi-Monte Carlo integration of digitally smooth functions by digital nets. In: Monte Carlo and Quasi-Monte Carlo Methods 1996 (Niederreiter, H. ed.), pp. 321–329. Lecture Notes in Statistics, Vol. 127, New York, Springer: 1997.

[4] Larcher, G., Schmid, W. Ch., Wolf, R.: Representation of functions as Walsh series to different bases and an application to the numerical integration of high-dimensional Walsh series. Math. Comp. **63**, 701–716 (1994).

[5] Larcher, G., Schmid, W. Ch., Wolf, R.: Quasi-Monte Carlo methods for the numerical integration of multivariate Walsh series. Math Comput. Modell. **23**, 55–67 (1996).

[6] Niederreiter, H.: Point sets and sequences with small discrepancy. Mh. Math. **104**, 273–337 (1987).

[7] Niederreiter, H., Xing, C.P.: Low-discrepancy sequences and global function fields with many rational places. Finite Fields Appl. **2**, 241–273 (1996).

[8] Pirsic, G.: Embedding theorems and numerical integration of Walsh series over groups. PhD thesis, University of Salzburg, September 1997.

**Author's address:** Dr. G. Pirsic, Institut für Diskrete Mathematik, Österreichische Akademie der Wissenschaften, Sonnenfelsgasse 19, A-1010 Wien, Austria.