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Sequential Compactness in Constructive Analysis

By

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Abstract

A new constructive notion of sequential compactness is introduced, and its relation to completeness and totally boundedness is explored.

In this note we complement the work in [3] by introducing, within the framework of (Bishop's) constructive mathematics [1], a new approach to sequential compactness. We begin with the fundamental definition on which the paper is based.

A sequence $\mathbf{x} = (x_n)$ in a metric space (X, ρ) **has at most one cluster point** if the following condition holds:

There exists $\delta_{\mathbf{x}} > 0$ such that if $0 < \delta < \delta_{\mathbf{x}}$ and $\rho(a, b) > 2\delta$, then either $\rho(x_n, a) > \delta$ for all sufficiently large n or else $\rho(x_n, b) > \delta$ for all sufficiently large n.

Note that each subsequence of (x_n) then has at most one cluster point: indeed, the same $\delta_{\mathbf{x}}$ works for such a subsequence as for the original sequence $\mathbf{x}$.

A Cauchy sequence $\mathbf{x}$ has at most one cluster point. To see this, let $\rho(a, b) > 2\delta > 0$. Choose $\varepsilon > 0$ such that $\rho(a, b) > 2(\delta + \varepsilon)$, and then choosen N such that $\rho(x_m, x_n) < \varepsilon$ for all $m, n \geq N$. Since

$$(\rho(x_N, a) - \delta - \varepsilon) + (\rho(x_N, b) - \delta - \varepsilon) \geq \rho(a, b) - 2(\delta + \varepsilon) > 0,$$

either $\rho(x_N, a) > \delta + \varepsilon$ or $\rho(x_N, b) > \delta + \varepsilon$. In the first case, $\rho(x_n, a) > \delta$ for all $n \geq N$; in the second, $\rho(x_n, b) > \delta$ for all $n \geq N$.

We call X **sequentially compact** if every sequence in X that has at most one cluster point converges to a limit in X. To see that this notion of sequential compactness is classically equivalent to the usual one,[1] suppose that X is sequentially compact in our sense, and let (x_n) be any sequence in X; if (x_n) does not have a cluster point, then it has at most one cluster point and so converges in X, a contradiction. On the other hand, suppose that X is sequentially compact in the usual sense, and consider a sequence (x_n) in X that has at most one cluster point. Since X is classically sequentially compact, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) that converges to a limit x_∞ in X. If (x_n) does not converge to x_∞, then there exists a subsequence of (x_n) that is bounded away from x_∞; this subsequence has cluster points, but none of those can equal x_∞; this contradicts our hypothesis that (x_n) has at most one cluster point.

Classically, a metric space is sequentially compact if and only if it is complete and totally bounded ([4], (3.16.1)). There is a natural approximate interval-halving proof that [0, 1] is constructively sequentially compact in our sense. Given a sequence (x_n) in [0, 1] that has at most one cluster point, let $I_0 = [0, 1]$. Taking $a = \frac{1}{5}$ and $b = \frac{4}{5}$ in the definition of *at most one cluster point*, we see that as $|a - b| > \frac{2}{5}$,

▷ either $|x_n - \frac{1}{5}| > \frac{1}{5}$, and therefore $x_n > \frac{2}{5}$, for all sufficiently large n;

▷ or else $|x_n - \frac{4}{5}| > \frac{1}{5}$, and therefore $x_n < \frac{3}{5}$, for all sufficiently large n.

In the first case, take $I_1 = \left[\frac{2}{5}, 1\right]$; in the second, take $I_1 = \left[0, \frac{3}{5}\right]$. Carrying on in this way, we produce closed intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that for each n, $|I_n| = \frac{3}{5}|I_{n-1} V$ and $x_k \in I_n$ for all sufficiently large k. Then there exists a unique point $x_\infty \in \bigcap_{n=0}^{\infty} I_n$, and it is routine to show that $x_\infty = \lim_{n\to\infty} x_n$.

The following key lemma will enable us to generalise this from [0, 1] to any complete, totally bounded metric space.

Lemma 1. *Let* $\mathbf{x} = (x_n)$ *be a sequence with at most one cluster point in a metric space* X*, let* $\delta_{\mathbf{x}}$ *be as in the foregoing definition, and let* $0 < \varepsilon < \delta_{\mathbf{x}}$. *Suppose that there exists a finitely enumerable*[2] *subset* F *of* X *such that for each* n *there exists* $x \in F$ *with* $\rho(x, x_n) < \varepsilon$. *Then* $\rho(x_m, x_n) < 8\varepsilon$ *for all sufficiently large* m *and* n.

[1] The classical property of sequential compactness does not hold constructively even for the pair set $\{0, 1\}$, and so is constructively useless.

[2] A set is **finitely enumerable** if it is the range of a mapping f from $\{1, \ldots, n\}$, for some natural number n. If also f is one – one, then its range is said to be **finite.**

Proof: Let $\xi_1 \in F$. Either $\rho(\xi, \xi_1) < 3\varepsilon$ for all $\xi \in F$ or else there exists $\xi' \in F$ such that $\rho(\xi', \xi_1) > 2\varepsilon$. In the first case we have $\rho(x_n, \xi_1) < 4\varepsilon$ for all n, and therefore $\rho(x_m, x_n) < 8\varepsilon$ for all m and n; so we may assume that the second case obtains. Accordingly, by our hypothesis on $\mathbf{x}$, either $\rho(x_n, \xi_1) > \varepsilon$ for all sufficiently large n or else $\rho(x_n, \xi') > \varepsilon$ for all sufficiently large n. Interchanging ξ_1 and ξ', if necessary, we may assume that $\rho(x_n, \xi_1) > \varepsilon$ for all $n \geq N_1$. If follows that for each $n \geq N_1$ there exists

$$\xi \in F \sim \{\xi_1\} = \{x \in F : x \neq \xi_1\}$$

such that $\rho(x_n, \xi) < \varepsilon$. We may therefore repeat the foregoing argument, with $\mathbf{x}$ replaced by $(x_n)_{n \geq N_1}$ and F replaced by $F \sim \{\xi_1\}$. In this way we obtain $\xi_2 \in F \sim \{\xi_1\}$ such that

- ▷ either $\rho(x_n, \xi_2) < 4\varepsilon$ for all $n \geq N_1$, and therefore $\rho(x_m, x_n) < 8\varepsilon$ for all $m, n \geq N_1$,
- ▷ or else there exists a positive integer $N_2 > N_1$ such that $\rho(x_n, \xi_2) > \varepsilon$ for all $n \geq N_2$.

Executing this procedure a total of at most $\#F$ times, we are guaranteed to produce N such that $\rho(x_m, x_n) < 8\varepsilon$ for all $m, n \geq N$. Q.E.D.

Corollary 2. *If X is a totally bounded metric space, then any sequence in X with at most one cluster point is a Cauchy sequence.*

Corollary 3. *The following are equivalent conditions on a sequence (x_n) in any metric space X:*

(i) (x_n) *is totally bounded and has at most one cluster point.*
(ii) (x_n) *is a Cauchy sequence.*

The following constructive generalisation of the Bolzano-Weierstraß Theorem is an immediate consequence of Corollary 2.

Theorem 4. *A complete, totally bounded metric space is sequentially compact.*

We now prove some partial converses of this theorem.

Proposition 5. *If X is sequentially compact, then it is complete.*

Proof: Every Cauchy sequence in X has at most one cluster point and so converges. Q.E.D.

Proposition 6. *Let X be sequentially compact, and let a be a point of X such that for all positives, t with $s < t$, either $\rho(x, a) < t$ for all $x \in X$ or else $\rho(x, a) > s$ for some $x \in X$. Then X is bounded.*

Proof: Construct an increasing binary sequence (λ_n) such that

▷ if $\lambda_n = 0$, then there exists $x \in X$ such that $\rho(x, a) > n$,

▷ if $\lambda_n = 1$, then $\rho(x, a) < n + 1$ for all $x \in X$.

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, choose $x_n \in X$ such that $\rho(x_n, a) > n$; if $\lambda_n = 1$, set $x_n = x_{n-1}$. To prove that $\mathbf{x} = (x_n)$ has at most one cluster point, let $\rho(y, z) > 2\delta > 0$, and choose a positive integer

$$N > \max\{\rho(a, y), \rho(a, z)\} + \delta.$$

If $\lambda_N = 1$, then $x_n = x_N$ for each $n \geq N$, so that either $\rho(x_n, y) > \delta$ for all $n \geq N$ or else $\rho(x_n, z) > \delta$ for all $n \geq N$. Consider, on the other hand, what happens if $\lambda_N = 0$. If $n \geq N$ and $\lambda_n = 0$, then $\rho(x_n, a) > n \geq N$, so

$$\rho(x_n, y) \geq \rho(x_n, a) - \rho(a, y) > \delta$$

and likewise $\rho(x_n, z) > \delta$. If $n \geq N$ and $\lambda_n = 1$, then there exists $k \in \{N + 1, \ldots, n\}$ such that $\lambda_k = 1 - \lambda_{k-1}$; whence $x_n = x_{n-1} = \cdots = x_{k-1}$ where, as above, $\rho(x_{k-1}, y) > \delta$ and $\rho(x_{k-1}, z) > \delta$.

Thus $\mathbf{x}$ has at most one cluster point in X and therefore converges to a limit $x_\infty \in X$. Choosing a positive integer $n > 1 + \rho(x_\infty, a)$ such that $\rho(x_n, x_\infty) < 1$, we see that $\lambda_n = 1$. Q.E.D.

The constructive **least-upper-bound principle** states that if the nonempty subset S of $\mathbf{R}$ is not only bounded above, but also **located** — in the sense that for all α, β with $\alpha < \beta$, either β is an upper bound of S or else there exists $x \in S$ with $x > \alpha$—then $\sup S$ exists. The locatedness condition cannot be dropped constructively, although it is redundant classically.

Corollary 7. *Under the hypotheses of* Proposition 6, $\sup_{x \in X} \rho(x, a)$ *exists.*

Proof: Since X is bounded by Proposition 6, we can apply the least-upper-bound principle to the set $\{\rho(x, a) : x \in X\}$. Q.E.D.

Proposition 8. *Let X be separable and sequentially compact. Then the following conditions are equivalent.*

(i) *For each* $\xi \in X$, $\sup_{x \in X} \rho(x, \xi)$ *exists.*

(ii) *X is totally bounded.*

Proof: Let $(a_n)_{n=1}^\infty$ be a dense sequence in X, and let $\varepsilon > 0$. Set $n_0 = 1$, assume (i), and construct an increasing binary sequence $(\lambda_k)_{k=1}^\infty$, and an increasing sequence $(n_k)_{k=1}^\infty$ of positive integers, such that

▷ if $\lambda_k = 0$, then $\rho(a_{n_k}, \{a_1, a_2, \ldots, a_{n_{k-1}}\}) > \varepsilon$,

▷ if $\lambda_k = 1$, then $\sup_{x \in X} \rho(x, \{a_1, a_2, \ldots, a_{n_{k-1}}\}) < 2\varepsilon$.

If $\lambda_k = 0$, put $x_k = a_{n_k}$; if $\lambda_k = 1$, put $x_k = x_{k-1}$. We show that the sequence $\mathbf{x} = (x_k)_{k=1}^{\infty}$ has at most one cluster point in X. To this end, let $0 < \delta < \varepsilon$ and $\rho(y, z) > 2\delta$, and choose j such that $\rho(y, a_j) < \varepsilon - \delta$. Either $\lambda_k = 1$ for some $k \leq j$, or else $\lambda_j = 0$. In the first case the sequence $\mathbf{x}$ is eventually constant and so clearly has at most one cluster point. In the second we may assume that $\lambda_{j+1} = 0$; so if $i \geq j + 1$ and $\lambda_i = 0$, then

$$\rho(y, x_i) = \rho(y, a_{n_i}) \geq \rho(a_j, a_{n_i}) - \rho(y, a_j) > \varepsilon - (\varepsilon - \delta) = \delta.$$

It follows that if $i > j + 1$ and $\lambda_i = 1$, then, as $x_i = x_k$ for some $k \in \{j + 1, \ldots, i - 1\}$ with $\lambda_k = 0$, we also have $\rho(y, x_i) > \delta$. This completes the proof that $\mathbf{x}$ has at most one cluster point.

Since X is sequentially compact, $\mathbf{x}$ converges to a limit $x_\infty \in X$. Choose κ such that $\rho(x_\infty, x_k) < \varepsilon/2$ for all $k \geq \kappa$. Then either $\lambda_\kappa = 1$ or else $\lambda_\kappa = 0$; in the latter case, as $\rho(x_{\kappa+1}, x_\kappa) < \varepsilon$, we must have $\lambda_{\kappa+1} = 1$. Hence $\{a_1, a_2, \ldots, a_{n_{\kappa+1}}\}$ is an ε-approximation to X. This completes the proof that (i) implies (ii).

If, conversely, (ii) holds, then the uniform continuity of the mapping $x \mapsto \rho(x, \xi)$ ensures that $\sup_{x \in X} \rho(x, \xi)$ exists ([1], page 94, (4.3)). Q.E.D.

It it tempting to try working with a simpler notion of "**x** has at most one cluster point": namely, that if *a, b* are distinct points of X, then either **x** is eventually bounded away from *a*, or **x** is eventually bounded away from *b*. However, Specker's Theorem ([5]; see also [2], page 58) shows that in the recursive model of constructive mathematics there exists a sequence in $[0, 1]$ which is eventually bounded away from *any* given recursive real number and, *a fortiori*, cannot converge.

References

[1] Bishop, E., Bridges, D.: Constructive analysis. Grundlehren der math. Wis-senschaften **279**. Heidelberg, Springer (1985).
[2] Bridges, D., Richman, F.: Varieties of constructive mathematics. London Math. Soc. Lecture Notes **95**. London, Cambridge Univ. Press (1987).
[3] Bridges, D., Ishihara, H., Schuster, P.: Compactness and continuity revisited. Preprint (1999).
[4] Dieudonné, J.: Foundations of modern analysis. New York: Academic Press (1960).
[5] Specker, E.: Nicht konstruktiv beweisbare Sätze der Analysis. J. Symbolic Logic **14**, 145–158 (1949).

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