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Congruence Classes Determining Congruence Kernels*

By

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Abstract

For an algebra $\mathfrak{A}$ with 0 we say that a congruence class C of $\Theta \in Con\mathfrak{A}$ determines the kernel of Θ if for any $\Phi \in Con\mathfrak{A}$ having the class C we have $[0]\Theta = [0]\Phi$. We characterize algebras satisfying this property. The characterization is useful in particular for lattices with 0. A result is given also for pseudocomplemented semilattices.

1. Preliminaries

Every algebra $\mathfrak{A}$ mentioned in the paper is supposed to have a constant term 0. Denote by $Con\mathfrak{A}$ the congruence lattice of $\mathfrak{A}$, and for $\Theta \in Con\mathfrak{A}$ the class $[0]\Theta$ is called the *kernel of* Θ. The aim of this paper is to describe those congruence classes of Θ which determine the kernel, more precisely:

Definition 1. Let $\Theta \in Con\mathfrak{A}$ and C be a class of Θ. We say that C *determines the kernel* of Θ if for every $\Phi \in Con\mathfrak{A}$ which has also the congruence class C the equality $[0]\Theta = [0]\Phi$ holds.

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Remark 1. If C determines the kernel of Θ, then C determines also the kernel of every $\Phi \in Con\mathfrak{A}$ having the class C.

At first, we recall some concepts connected with this definition.

An algebra $\mathfrak{A}$ is *regular* if for each $\Theta, \Phi \in Con\mathfrak{A}$ we have $\Theta = \Phi$ whenever they have a congruence class in common. $\mathfrak{A}$ is *weakly regular* if $[0]\Theta = [0]\Phi$ implies $\Theta = \Phi$ for every $\Theta, \Phi \in Con\mathfrak{A}$. In [3], the concept of local regularity was introduced: $\mathfrak{A}$ is *locally regular* if for each $\Theta, \Phi \in Con\mathfrak{A}$ we have $[0]\Theta = [0]\Phi$ whenever Θ, Φ have a congruence class in common. The paper [3] contains characterizations of varieties of locally regular algebras.

Remark 2. a) $\mathfrak{A}$ is regular if and only if $\mathfrak{A}$ is both weakly regular and locally regular.

b) $\mathfrak{A}$ is locally regular if and only if every congruence class C of each $\Theta \in Con\mathfrak{A}$ determines the kernel of Θ.

Example 1. (see [4]). Every complemented lattice is locally regular (but not necessary regular). Every relatively complemented lattice with 0 is regular (this is a well-known result of J. Hashimoto [5]).

Let us mention that regular lattices were characterized in [2]. An analogous method was used by O. M. Mamedov [6] to characterize regular algebras in the general case. We are going to apply a similar approach to our problem.

If M is a subset of A, then by $\Theta(M)$ we denote the least congruence on $\mathfrak{A}$ containing $M \times M$.

2. A Characterization of Classes Determining Kernels

Theorem 1. *Let $\mathfrak{A}$ be an algebra with $0, \Theta \in Con\mathfrak{A}$ and C be a class of Θ. Then C determines the kernel of Θ if and only if the following condition holds for every $\Phi \in Con\mathfrak{A}$ having the class C:*

$$x \in [0]\Phi \text{ if and only if there exist } c_1, \ldots, c_n \in C \text{ such that } x\Theta(c_1, \ldots, c_n)0. \qquad (*)$$

Proof:

a) Suppose that C determines the kernel of Θ, then C determines the kernel of every $\Phi \in Con\mathfrak{A}$ having the class C. Let $\Psi = \Theta(C)$ then of course, C is also a class of Ψ and $\Psi \subseteq \Phi$. Since Φ, Ψ have the class C in common, we have $[0]\Phi = [0]\Psi$.

If $x \in [0]\Phi$ then $x \Psi 0$ whence $x[\bigvee\{\Theta(c, c'); (c, c') \in C \times C\}]0$. Since $Con\mathfrak{A}$ is an algebraic lattice, there exists a finite subset $\{c_1, \ldots, c_n\} \subseteq C$ with $x\Theta(c_1, \ldots, c_n)0$. Conversely, if $x\Theta(c_1, \ldots, c_n)0$

for some $\{c_1, \dots, c_n\} \subseteq C$ then $x \Psi 0$, hence $x \Phi 0$ giving $x \in [0]\Phi$. Thus, the condition $(*)$ holds.

b) Suppose that $(*)$ holds for each $x \in A$. Suppose that $x \in [0]\Theta$ and that $\Phi \in Con\mathfrak{A}$ has also the class C. Then $\Theta(c_1, \dots, c_n) \subseteq \Phi$ for every $c_1, \dots, c_n \in C$ and hence $x \Phi 0$, i.e. $x \in [0]\Phi$. We have shown $[0]\Theta \subseteq [0]\Phi$. Interchanging the roles of Θ, Φ, we can prove the converse inclusion, thus $[0]\Theta = [0]\Phi$ proving that C determines the kernel of Θ. □

For lattices, the situation is a bit more simple:

Theorem 2. *Let $\mathfrak{L}$ be a lattice with $0, \Theta \in Con\mathfrak{L}$ and C be a class of Θ. Then C determines the kernel of Θ if and only if the following condition holds for every $\Phi \in Con\mathfrak{L}$ having the class C:*

$$x \in [0]\Phi \text{ if and only if there exist } c, d \in C \text{ with } x\Theta(c,d)0. \quad (**)$$

Proof: If $(**)$ holds then, by Theorem 1, C determines the kernel of Θ. Conversely, if C determines the kernel of Θ and $\Phi \in Con\mathfrak{L}$ has the class C then, by Theorem 1, for each $x \in [0]\Phi$ there exist $c_1, \dots, c_n \in C$ such that $(*)$ holds. Take $c = c_1 \wedge \dots \wedge c_n$ and $d = c_1 \vee \dots \vee c_n$. Of course, $c\Theta(c_1, \dots, c_n)d$ whence $\Theta(c,d) \subseteq \Theta(c_1, \dots, c_n)$.

On the other hand, $c \le c_i \le d$ for $i = 1, \dots, n$ thus $c_i\Theta(c,d)c_j$ for $i, j \in \{1, \dots, n\}$ and hence $\Theta(c_1, \dots, c_n) = \Theta(c,d)$. Altogether, $(**)$ is proved. □

3. A Characterization of Locally Regular Algebras

Theorem 3. *An algebra $\mathfrak{A}$ with 0 is locally regular if and only if for every $x, y \in A$ there exist $c_1, \dots, c_n \in A$ such that $\Theta(0,x) = \Theta(y, c_1, \dots, c_n)$.*

Proof: Assume that for every $x, y \in A$ there exist $c_1, \dots, c_n \in A$ such that $\Theta(0,x) = \Theta(y, c_1, \dots, c_n)$. In the following, we use Remark 2b. So let $\Phi, \Psi \in Con\mathfrak{A}$ and suppose that they have the class C in common.

Let $y \in C$, i.e. $[y]\Phi = [y]\Psi$. Take $x \in [0]\Phi$. Then $x \Phi 0$, i.e. $\Theta(y, c_1, \dots, c_n) = \Theta(0,x) \subseteq \Phi$ whence $c_1, \dots, c_n \in [y]\Phi = [y]\Psi$. Thus $\Theta(y, c_1, \dots, c_n) \subseteq \Psi$ giving $x \in [0]\Psi$. We have shown $[0]\Phi \subseteq [0]\Psi$. The converse inclusion can be shown analogously, hence $\mathfrak{A}$ is locally regular.

Conversely, let $\mathfrak{A}$ be locally regular, $x, y \in A$ and $C = [y]\Theta(0,x)$. Let $\Psi = \Theta(C)$, then C is a class of Ψ and $\Psi \subseteq \Theta(0,x)$. Since $\mathfrak{A}$ is

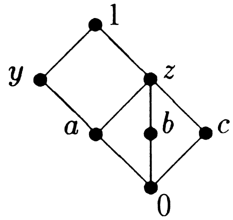

Fig. 1

locally regular, this yields $[0]\Theta(0,x) = [0]\Psi$, i.e. $0\,\Psi\,x$, hence $0[\bigvee\{\Theta(y,c); c \in C\}]x$.

Since $Con\mathfrak{A}$ is an algebraic lattice, there exists a finite subset $\{c_1, \ldots, c_n\} \subseteq C$ such that $0\Theta(y, c_1, \ldots, c_n)x$ whence $\Theta(0,x) \subseteq \Theta(y, c_1, \ldots, c_n) \subseteq \Psi$, and $\Psi \subseteq \Theta(0,x)$ implies $\Theta(0,x) = \Theta(y, c_1, \ldots, c_n)$. □

By the same lines as in the proof of Theorem 2, we can conclude immediately:

Theorem 4. *A lattice $\mathfrak{L}$ with* 0 *is locally regular if and only if for each $x, y \in L$ there exist $c, d \in L$ such that $\Theta(0,x) = \Theta(y,c,d)$.*

Example 2. Let $\mathfrak{L}$ be the lattice depicted in Fig. 1.

Then $Con\mathfrak{L} = \{\omega, \Phi, \Psi, L \times L\}$, where ω is the identical congruence of $\mathfrak{L}$ and Φ, Ψ are given by their partitions:

$$\Phi \ldots \{a,y\}, \{1,z\}, \{0\}, \{b\}, \{c\}$$
$$\Psi \ldots \{y,1\}, \{0,a,b,c,z\}.$$

Clearly, the class $C = [y]\Psi$ determines the kernel of Ψ and every class of Φ determines the kernel $[0]\Phi = \{0\}$. So we recognize immediately that $\mathfrak{L}$ is locally regular. On the contrary, $\mathfrak{L}$ is not regular since it is not weakly regular: namely $[0]\Phi = \{0\} = [0]\omega$ but $[y]\Phi = \{y,a\} \neq [y]\omega$.

Remark 3. Conditions similar to that of Theorem 3 were studied in [1] and [6] under the names (zero) congruence (n-)transferability.

4. Classes Determining Kernels in Lattices with 0 and Pseudocomplemented Semilattices

We will now discuss another approach which is tailored especially for lattices and pseudocomplemented semilattices.

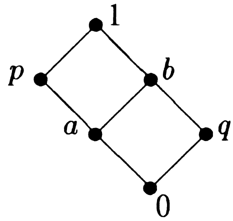

Fig. 2

Theorem 5. *Let $\mathfrak{L}$ be a lattice with $0, \Theta \in Con\mathfrak{L}$ and C be a class of Θ. If for each $\Phi \in Con\mathfrak{A}$ having the class C and each $z \in [0]\Phi$ there exists an $x \in C$ with $z \wedge x = 0$ then C determines the kernel of Θ.*

Proof: Let C be a common class of $\Theta, \Phi \in Con\mathfrak{L}$ and let $z \in [0]\Theta$. Suppose there exists an $x \in C$ with $z \wedge x = 0$. Since $z \in [0]\Theta$, we have $0\,\Theta\, z$ and $x = x \vee 0\,\Theta\, x \vee z$, i.e. also $x \vee z \in C$ whence $x\,\Phi\, x \vee z$. Thus $0 = x \wedge z\Phi(x \vee z) \wedge z = z$ proving $z \in [0]\Phi$, i.e. $[0]\Theta \subseteq [0]\Phi$. The converse inclusion can be shown analogously, thus $[0]\Theta = [0]\Phi$ and hence C determines the kernel of Θ. □

Remark 4. If the assumption of Theorem 5 does not hold then C does not need to determine the kernel of Θ, see the following

Example 3. Let $\mathfrak{L}$ be the lattice in Fig. 2.

The congruences $\Theta, \Phi \in Con\mathfrak{L}$ are given by the partitions

$$\Theta \ldots \{0, a\}, \{b, q\}, \{p\}, \{1\}$$
$$\Phi \ldots \{a, p\}, \{b, 1\}, \{0\}, \{q\}.$$

Again, ω denotes the identical congruence. It is easy to see that every class of Φ determines the kernel of Φ.

On the contrary, the class $[p]\Theta$ or $[1]\Theta$ of Θ does not determine the kernel of Θ since $[p]\Theta = [p]\omega$, $[1]\Theta = [1]\omega$ but $[0]\Theta = \{0, a\} \neq [0]\omega$. The assumption of Theorem 5 does not hold since $a \wedge p = a \neq 0$ and $a \wedge 1 = a \neq 0$.

Furthermore, let us note that the class $[b]\Theta$ determines the kernel of Θ.

Now, we will solve our problem for pseudocomplemented semilattices. Let us note that a particular case was solved by J. C. Varlet in [7] and a more general solution was presented by the authors in [4]. Namely, it was shown in [4] that a class C of $\Theta \in Con\mathfrak{S}$ ($\mathfrak{S}$ being a pseudocomplemented semilattice) determines the kernel of Θ if C is "large enough", i.e. if for some $x \in C$ also $x^{**} \in C$.

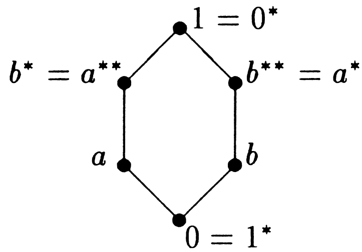

Fig. 3

Theorem 6. *Let* $\mathfrak{S} = (S; \wedge,^*, 0)$ *be a pseudocomplemented* $\wedge$-*semilattice,* $\Theta \in Con\mathfrak{S}$ *and* C *be a class of* Θ. *If for each* $\Phi \in Con\mathfrak{A}$ *having the class* C *and each* $z \in [0]\Phi$ *there exists an* $x \in S$ *such that* $x^{**} \in C$ *and* $z \wedge x^{**} = 0$ *then* C *determines the kernel of* Θ.

Proof: Let C be a common class of $\Theta, \Phi \in Con\mathfrak{S}$ and let $z \in [0]\Theta$. By our assumption, there exists an $x \in S$ such that $x^{**} \in C$ and $z \wedge x^{**} = 0$. Since $z \in [0]\Theta$ we have $0\,\Theta\, z$ and hence $x^{**} = (x^* \wedge 0^*)^*\Theta(x^* \wedge z^*)^*$ thus also $(x^* \wedge z^*)^* \in C$. This yields $x^{**}\Phi(x^* \wedge z^*)^*$.

Of course, $x^* \wedge z^* \le z^*$ thus $(x^* \wedge z^*)^* \ge z^{**} \ge z$ which yields $(x^* \wedge z^*)^* \wedge z = z$. Using this, we obtain $0 = x^{**} \wedge z\Phi(x^* \wedge z^*)^* \wedge z = z$ whence $z \in [0]\Phi$.

We have shown $[0]\Theta \subseteq [0]\Phi$. The converse inclusion can be proved analogously, i.e. C determines the kernel of Θ. □

Example 4. Let $\mathfrak{S}$ be the pseudocomplemented $\wedge$-semilattice depicted in Fig. 3 and $\Theta \in Con\mathfrak{S}$ defined by the classes $[0]\Theta = \{0, b, b^{**}\}, [a]\Theta = \{a\}, [b^*]\Theta = \{b^*, 1\}$.

Then it is easy to see that the class $C = [b^*]\Theta$ determines the kernel of Θ. On the contrary, the class $[a]\Theta$ does not determine the kernel of Θ since $[a]\Theta = \{a\} = [a]\omega$ but $[0]\Theta = \{0, b, b^{**}\} \neq \{0\} = [0]\omega$, where ω denotes the identical congruence of $\mathfrak{S}$.

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