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A Conditional Gołąb–Schinzel Equation

Von

M. Sablik

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durch das w. M. Ludwig Reich)

Abstract

We determine the general continuous solution of Gołąb–Schinzel functional equation assumed to hold on an interval containing 0.

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1. Introduction

In the present paper we are going to present continuous solutions of a conditional Gołąb–Schinzel equation. More exactly we are interested in solving the following equation

$$f(x + yf(x)) = f(x)f(y), \tag{GS}$$

which holds for every x, y from a real interval I containing 0, and the unknown continuous function f is defined on a real interval

$$I_f = \{x + yf(x) : x, y \in I\}.$$

Analogous questions for some specific intervals I were treated by J. Aczél and J. Schwaiger in [1], and by L. Reich in [3] and [4].

Note that a part of our work is to determine intervals I and I_f. Let us also observe that obviously $I \subset I_f$ and therefore (GS) is not an empty condition.

As a main tool in our considerations we will use some of our results from [5]. That paper concerns the following generalization of a functional equation of N. Abel

$$\psi(xh(y) + yg(x)) = \varphi(x) + \varphi(y), \quad x \in I, \tag{A}$$

where all the four functions are unknown, as well as the real interval I which is assumed to satisfy only the requirement $0 \in I$. Solving (A) we reduced it to a slightly less general equation

$$\Psi(xF(y) + yG(x)) = \Psi(x) + \Psi(y) \tag{A$'$}$$

and in particular we obtained the following (cf. [5, Propositions 2.7 and 2.9]).

Proposition 1.1. *If* (Ψ, F, G) *is a continuous solution of* (A') *which satisfies*

$$\Psi(0) = 0, \quad F(0) = 1, \quad G(0) \neq 0,$$

then there exists a $c \in \mathbb{R}$ *such that*

$$G(x) = F(x) + cx \tag{1.1}$$

for every $x \in I$.

2. Gołąb–Schinzel Equation

If we set $x = y = 0$ in (GS) then we get $f(0) \in \{0, 1\}$. The case $f(0) = 0$ yields the trivial solution $f = 0$ which is easily seen when we put $y = 0$ into (GS). Let us assume therefore that $f(0) = 1$. Then by continuity, f is positive in an interval J containing 0. It follows from (GS) that also its lefthand side is positive whenever $x, y \in J$. Thus, if $x, y \in J$, we can take logarithms of both sides of (GS). Denoting $\Psi := \ln f$, $F := 1$, and $G = f$ we see that (GS) restricted to J implies (A') for the above defined functions, which holds for all $x, y \in J$. We can make use of Proposition 1.1 and thus from (1.1) we infer that there exists a constant $c \in \mathbb{R}$ such that

$$f(x) = 1 + cx \tag{2.1}$$

holds for all $x \in J$.

Actually what we have proved is that f is given by (2.1) on any interval containing 0 and such that f is positive on this interval. In particular, if we look for those solutions of (GS) which are positive on I then we see that they are given by (2.1). Obviously, in this case $I \subset \bar{c}(-\infty, 1)$ if $c \neq 0$. Here and in the sequel $\bar{c}$ stands for $\frac{-1}{c}$. We have also

$$I_f = \{x + y + cxy : x, y \in I\} \tag{2.2}$$

and $I_f \subset \overline{c}(-\infty, 1)$. Moreover, it easily follows from (GS) that f is given by (2.1) on I_f.

Suppose now that $0 \in f(I)$. Then $c \neq 0$, and $\overline{c} \in I$, and f is given by (2.1) on $I \cap (\overline{c}(-\infty, 1))$. Let us define $\gamma : I \to \mathbb{R}$ by

$$\gamma(x) = x + \overline{c}f(x).$$

Since γ is continuous, we see that $U = \gamma(I)$ is an interval. Consider two cases.

A. U is a degenerate interval. Then $U = \{\overline{c}\}$ because $\overline{c} = \gamma(\overline{c})$. This means that

$$x + \overline{c}f(x) = \overline{c}$$

for every $x \in I$, or f is given by (2.1) in I. One easily notes that I_f is given by (2.2), and f is given by (2.1) in I_f.

B. U is a non-degenerate interval. Without loss of generality assume that $c < 0$. As we have seen above, $\overline{c} \in U$. On the other hand

$$f(\gamma(x)) = f(x)f(\overline{c}) = 0$$

so

$$f|U = 0. \tag{2.3}$$

Therefore, because f is given by (2.1) at the lefthand side of $\overline{c}$, the interval U has the form $U = [\overline{c}, y_1]$ or $U = [\overline{c}, y_1)$ for some $y_1 \in (\overline{c}, \infty]$. Put

$$y_2 = \sup\{y_1 > \overline{c} : f|[\overline{c}, y_1] = 0\}.$$

We will show that $y_2 \geq \sup I$. Indeed, suppose to the contrary that $y_2 < \sup I$. Then by continuity of f we have $f|[\overline{c}, y_2] = 0$. Consider the interval

$$V = \{x + y_2 f(x) : x \in I\}.$$

From (GS) we infer that $f|V = 0$. Moreover, $y_2 = 0 + y_2 f(0)$ and $\overline{c} = \overline{c} + y_2 f(\overline{c})$ belong to V, whence we infer that $V = [\overline{c}, y_2]$. From the definition of V we get therefore

$$f(x) < 0 \tag{2.4}$$

for every $x \in I \cap (y_2, \infty)$. Fix an $x_0 \in I \cap (y_2, \infty)$ and consider the function $\varphi : [0, \overline{c}] \to \mathbb{R}$ given by

$$\varphi(z) = z + x_0 f(z) = x_0 + (1 + cx_0)z.$$

Thus φ is an affine function and since $f(0) = x_0$ and $f(\overline{c}) = \overline{c}$, we obtain

$$\varphi([0, \overline{c}]) = [\overline{c}, x_0].$$

In particular there exists a $z \in (0, \overline{c})$ such that $\varphi(z) = y_2$. From (GS), (2.4) and the already established form of f in $I \cap (-\infty, \overline{c})$ we get

$$0 = f(y_2) = f(\varphi(z)) = f(z)f(x_0) < 0.$$

This contradiction shows that $f|[\overline{c}, \infty) = 0$. Thus $f|I$ is given by

$$f(x) = \max\{1 + cx, 0\} \tag{2.5}$$

for every $x \in I$. As a simple consequence of (GS) we obtain that (2.5) holds also for any $x \in I_f = \{x + yf(x) : x \in I\}$.

Let us summarize the above considerations in the following.

Theorem 2.1. *Let I be a real interval containing 0. A continuous function $f : I_f := \{x + yf(x) : x, y \in I\} \to \mathbb{R}$ satisfies (GS) for every $x, y \in I$ if and only if $f = 0$ or there exists a constant $c \in \mathbb{R}$ such that f is given by (2.1) or by (2.5).*

Remark 2.2. If we take $I = [0, \infty)$ then it follows from the above theorem that nontrivial continuous solutions of (GS) which holds for nonnegative x and y are given either by (2.5) or by (2.1). In the former case $c \geq 0$ and $I_f = [0, \infty)$. In the latter one $c < 0$ is admitted and, if this is the case, then $I_f = \mathbb{R}$, as it can be seen easily.

Remark 2.3. Suppose that we are looking for continuous solutions of (GS) which holds for $x, y \in [0, \infty)$ assuming moreover that they fulfill the condition $x + yf(x) \geq 0$ for every $x, y \in [0, \infty)$. Then from the above theorem it follows that either $f = 0$ or f is given by (2.1) with $c \geq 0$ or f is given by (2.5) with $c < 0$. In both cases we have $I_f = [0, \infty)$.

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Author's address: Prof. Dr. M. Sablik, Institute of Mathematics, Silesian University, Bankowa 14, 40 007 Katowice, Poland.