VIENNA INSTITUTE OF DEMOGRAPHY

Working Papers

8 / 2009

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# *Population Ageing and Endogenous Economic Growth*

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## Abstract

This article investigates the consequences of population ageing for long-run economic growth perspectives. We introduce population ageing into a generalized model of endogenous technological change incorporating the model of Romer (1990) and Jones (1995) as special cases. We find that increases in longevity have positive effects on steady state per capita output growth in endogenous as well as in semi-endogenous growth models. In the latter case, the positive dependence can also be shown for the equilibrium growth rate during transition to the steady state.

JEL classification: O41, J10, C61

## Keywords

Population ageing, endogenous technological change, long-run economic growth

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## Acknowledgements

I would like to thank Dalkhat Ediev (Vienna Institute of Demography), Theresa Grafeneder-Weissteiner (Vienna University of Economics and Business), Ingrid Kubin (Vienna University of Economics and Business), Alexia Prskawetz (Vienna University of Technology), Holger Strulik (University of Hannover), Vladimir Veliov (Vienna University of Technology) and Stefan Wrzaczek (Vienna University of Technology) for useful comments and suggestions. Financial support by the Vienna Science and Technology Fund (WWTF) in its ``Mathematics and..." call 2007 is greatly acknowledged.

# Population Ageing and Endogenous Economic Growth

Klaus Prettner

## 1 Introduction

Most recently, population ageing in industrialized countries was identified to be one of the central topics regarding future economic development (United Nations (2007), Eurostat (2009), The Economist (2009)). Its consequences are expected to be huge. To mention only the most well known examples: support ratios will decline such that fewer and fewer workers have to carry the burden of financing more and more retirees (see for example Gertler (1999) and Gruescu (2007)); overall productivity levels will change because individual workers have age specific productivity profiles (see Skirbekk (2008) for an overview); the savings behaviour of individuals will change because they expect to live longer (see for example Futagami and Nakajima (2001)). However, as regards the implications of population ageing on per capita output growth, there are only transient effects of changing support ratios, changing saving behaviour of households and changing aggregate productivity profiles. The reason is that on the one hand, a one time shift from high to low fertility cannot lead to a permanently changing age decomposition of a certain population and on the other hand, changes in the savings behaviour of households have only level effects on per capita output (Ramsey (1928), Solow (1956)).

In this paper we concentrate on the implications of population ageing on per capita output *growth* over a long time horizon. Since technological progress has been identified as main driving force behind economic development (see for example Romer (1990)), we are particularly interested in the effects of changing age decompositions on research and development (R & D). Therefore the natural model class to examine our research question are endogenous and semi-endogenous growth models, where the research effort is determined in a general equilibrium framework assuming utility maximizing households and profit maximizing firms.

Endogenous growth models (see for example Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992)) state that, aside from other influences, the population size of a certain country is crucial for long-run economic performance. Larger countries are able to grow faster because they have more scientists to employ and they have a larger market such that profit opportunities of firms engaging in R & D are larger. This effect is called the scale effect which was questioned by Jones (1995) because it is not supported by empirical evidence. In setting up a scale-free model of technological change, Jones (1995) paved the way for semi-endogenous growth models (see also Kortum (1997) and Segerström (1999)), where long-run economic performance is affected by population *growth* rather than population *size*. The basic idea of semi-endogenous growth models is that research becomes more and more complex with an increasing level of technology. Consequently, ever more resources have to be devoted to it in order to sustain a certain pace of development.

Although the described models examine the effects of changes in demographic patterns as represented by population size and population growth, they remain silent when it comes to the consequences of population ageing because they assume that people live forever. We introduce age dependent heterogeneity of individuals into these models by generalizing them to account for finite planning horizons and overlapping generations in the spirit of Blanchard (1985) and Buiter (1988). In doing so we assume that individuals do not live forever but that they have to face a certain probability of death at each instant. The standard endogenous and semi-endogenous growth models are then special cases, where the probability of death is equal to zero.

The paper proceeds as follows: Section 2 describes a model that nests the Romer (1990) and the Jones (1995) framework. We derive equilibrium and steady state growth rates in both cases. Section 3 introduces population ageing and examines its effects in both types of models. Finally, section 4 draws conclusions and highlights scope for further research.

# 2 The basic model

This section characterizes the basic model of R & D which relies on horizontal innovations, i.e. on the development of new product varieties[1]. It nests the Romer (1990) framework with strong spillovers in the research sector and a constant population size as well as the Jones (1995) framework with weaker spillovers in the research sector and a growing population size as special cases (see also Strulik (2009)). We assume that time evolves continuously and that individuals have infinite planning horizons. In section 3, when we examine the effects of population ageing, we will

[1]Using a model with vertical innovations would not change the results.

replace the latter assumption and introduce an overlapping generations structure. There are three sectors: final goods production, intermediate goods production and R & D. The economy has two productive factors at its disposal: capital and labour. Labour and intermediates are used to produce final goods, capital and blueprints are used in intermediate goods production and labour is used to produce blueprints in the R & D sector. Furthermore, as it is standard in these types of models, we assume perfect competition in the final goods sector and in the research sector, whereas there is monopolistic competition in the intermediate goods sector.

The discussion in this section builds on Romer (1990) and Jones (1995) with some slight modifications. First of all, we do not assume that there is only one single representative individual who maximizes its discounted stream of lifetime utility. Instead, $L$ identical individuals are contemporaneously living at each point in time $t$. The reason for this assumption is that the model can then be consistently generalized to allow for a changing age decomposition of the population. Furthermore, the differences between the model where people live forever and the model where they have to face a constant risk of death can be highlighted more explicitly. Secondly, in contrast to Jones (1995) but without loss of generality, we do not allow for duplication in the R & D process. The reason is that with this simplification Romer (1990) and Jones (1995) are special cases of a more general approach.

## 2.1 Consumption side

Suppressing time subscripts, a certain individual maximizes its discounted stream of lifetime utility

$$U = \int_{t_0}^{\infty} e^{-\rho(\tau - t_0)} \left( \frac{c^{1-\sigma} - 1}{1 - \sigma} \right) d\tau, \tag{1}$$

where $t_0$ is the date of birth of the individual, i.e. the starting point of economic activities in the respective country, $\rho$ is the subjective discount rate, $c$ refers to individual consumption of the final good and $\sigma$ is a coefficient of relative risk aversion such that the intertemporal elasticity of substitution is $1/\sigma$. The wealth constraint of each individual reads

$$\dot{k} = (r - \delta)k + wl - c, \tag{2}$$

where $k$ refers to the individual capital stock, $r$ is the rate of return on capital, $\delta$ is the rate of depreciation, $w$ represents the wage rate and $l$ refers to the efficiency units of labour an individual supplies on the labour market. For simplicity we take the normalization $l \equiv 1$. Carrying out utility maximization subject to the wealth constraint yields the familiar individual Euler equation

$$\dot{c} \;=\; \frac{(r-\rho-\delta)}{\sigma}c. \tag{3}$$

Since our economy is not featuring one single representative individual but a stationary population consisting of a large number of individuals, we define uppercase letters as population aggregates and write the aggregate law of motion for capital and the "aggregate" Euler equation as

$$\dot{K} \;=\; (r-\delta)K + wL - C, \tag{4}$$

$$\dot{C} \;=\; \frac{(r-\rho-\delta)}{\sigma}C, \tag{5}$$

where $L$ refers to the population size being equivalent to the cohort size $N$ for obvious reasons. Due to the fact that there is no heterogeneity of individuals with respect to age, aggregate equations do not differ from individual equations in the sense that growth rates of capital and consumption are similar at the individual level and economy-wide.

In case of the Jones (1995) model with population growth, we follow Acemoglu (2009) who augments the discount rate by the population growth rate $n$, such that the individual Euler equation becomes

$$\frac{\dot{c}}{c} = \frac{(r-\rho-n-\delta)}{\sigma}. \tag{6}$$

Without age specific heterogeneity the aggregate Euler equation is then also represented by equation (5).

## 2.2 Production side

The final goods sector produces the consumption good (numeraire) with labour and intermediates as inputs. To have a sensible economic interpretation, one can refer to intermediate varieties as different machines. Consequently, the final goods sector produces with a technology of the form

$$Y = L_Y^{1-\alpha}\int_0^A x_i^\alpha di, \tag{7}$$

where $Y$ represents output of the consumption good, $L_Y$ refers to labour used in final goods production, $A$ is the technological frontier, i.e. the number of different machines available, $x_i$ is the amount of a certain specific machine used in final goods production and $\alpha$ is the intermediate share. Profit maximization and the assumption of perfect competition in the final goods sector imply that factors are paid their marginal products:

$$\begin{aligned} w_Y &= (1-\alpha)\frac{Y}{L_Y}, && (8) \\ p_i &= \alpha L_Y^{1-\alpha} x_i^{\alpha-1}. && (9) \end{aligned}$$

Here $w_Y$ refers to the wage rate paid in the final goods sector and $p_i$ to prices paid for intermediate inputs. Note that all specific machines are used to the same extent so the index $i$ can be dropped due to symmetry.

The intermediate goods sector is monopolistically competitive in the spirit of Dixit and Stiglitz (1977) and each firm produces one of the specific machines. In order to do so, it has to purchase one blueprint from the R & D sector and afterwards employ capital as variable input in production. The costs of blueprints represent fixed costs for each firm. Free entry will ensure that operable profits equal fixed costs such that overall profits are zero[2]. After an intermediate goods producer has purchased a blueprint, it can transform one unit of capital into one unit of the intermediate good. Thus operating profits can be written as

$$\pi = p(x)k - rk = \alpha L_y^{1-\alpha} k^{\alpha} - rk. \tag{10}$$

Profit maximization of firms yields prices of machines

$$p = \frac{r}{\alpha}, \tag{11}$$

where $1/\alpha$ is the markup over marginal costs (see also Dixit and Stiglitz (1977)). The aggregate capital stock is equal to the number of all intermediates produced, i.e. $K = Ax$, such that equation (7) becomes

$$Y = (AL_Y)^{1-\alpha} K^{\alpha}. \tag{12}$$

The R & D sector employs scientists to discover new blueprints. Depending on the productivity of scientists, $\lambda$, and the size of technology spillovers, $\phi$, the number of blueprints evolves according to

$$\dot{A} = \lambda A^{\phi} L_A, \tag{13}$$

[2]If positive overall profits were present, new firms would enter the market until these profits are vanished.

where $L_A$ denotes the number of scientists employed. Consequently, the technological frontier expands faster if scientists are more productive or technological spillovers are higher. If $\phi = 1$, spillovers are strong enough and developing new blueprints does not become ever more difficult as the technological frontier expands. If in contrast $\phi < 1$, the spillovers are insufficiently low and developing new blueprints becomes more and more difficult with an expanding technological frontier. In the former case our economy behaves like in the Romer (1990) scenario, whereas in the latter case our economy behaves like in the Jones (1995) scenario. Furthermore, there is perfect competition in the research sector such that firms maximize

$$\max_{L_A} \pi_A = p_A \lambda A^{\phi} L_A - w_A L_A, \tag{14}$$

with $\pi_A$ being the profit of a firm in the R & D sector and $p_A$ representing the price of a blueprint. The first order condition pins down wages in the research sector to

$$w_A = p_A \lambda A^{\phi}. \tag{15}$$

## 2.3 Market clearing

There is perfect labour mobility between sectors, therefore wages of final goods producers and wages of scientists equalize. The reason is that workers in the final goods sector and scientists do not differ with respect to education or with respect to productivity. Consequently, if wages were higher in one of these two sectors, it would attract workers from the other sector until wages equalize again. Therefore we can insert (8) into (15) to get to following equilibrium condition:

$$p_A \lambda A^{\phi} = (1-\alpha)\frac{Y}{L_Y}. \tag{16}$$

Firms in the R & D sector can charge prices of blueprints that are equal to the present value of operating profits in the intermediate goods sector because there is always a potential entrant who is willing to pay this price. Therefore we have

$$p_A = \int_{t_0}^{\infty} e^{-(R(\tau)-R(t_0))} \pi \, d\tau, \tag{17}$$

where $R(t_0) = \int_0^{t_0} (r(s) - \delta) \, ds$, i.e. the discount rate is the market interest rate paid for household's savings. Via the Leibniz rule and the fact that prices of blueprints do not change in the steady state, we can obtain

$$p_A = \frac{\pi}{r - \delta} \tag{18}$$

such that these prices are equal to operating profits of intermediate goods producers divided by the market interest rate[3]. Next, we obtain profits by using equation (10) as

$$\pi = (1 - \alpha)\alpha \frac{Y}{A} \tag{19}$$

such that equation (18) becomes

$$p_A = \frac{(1 - \alpha)\alpha Y}{(r - \delta) A}. \tag{20}$$

Assuming that labour markets clear, i.e. $L = L_A + L_Y$, we can determine the fraction of workers employed in the final goods sector and in the R & D sector by using equation (16):

$$\begin{aligned} L_Y &= \frac{(r - \delta) A^{1-\phi}}{\alpha \lambda}, \\ L_A &= L - \frac{(r - \delta) A^{1-\phi}}{\alpha \lambda}. \end{aligned} \tag{21}$$

The interpretation of these two equations is straightforward: the higher the market interest rate on capital, $r - \delta$, the higher are the opportunity costs of R & D and consequently, the lower is the number of scientists and the higher is the number final goods assemblers employed; the higher the productivity of researchers, $\lambda$, the more scientists and the less assemblers of final goods are employed; if knowledge spillovers $\phi$ are insufficiently low to prevent R & D to become ever more complex, an expanding technological frontier $A$ reduces employment of scientists and increases employment of workers in the final goods sector; finally, an increase in the intermediate share of final output, $\alpha$, increases the number of scientists and decreases the number of workers in the final goods sector because it raises operating profits in the intermediate sector and therefore prices of blueprints. Inserting (21) into (13) leads to the evolution of knowledge:

$$\dot{A} = \lambda A^{\phi} L - \frac{(r - \delta) A}{\alpha}, \tag{22}$$

[3]Note that we cannot analyse transition dynamics to an optimal capital stock in this case. Instead, we immediately jump to the optimal capital stock.

where we see that the technological frontier expends faster, the larger the population size is. All factors identified above to reduce the number of scientists employed in the R & D sector also reduce the pace of technological progress. From now on we have to distinguish between the Romer (1990) case, where technological spillovers are strong and the population size is constant, and the Jones (1995) case, where technological spillovers are weaker and the population grows at rate $n$.

## 2.4 The Romer (1990) case

After implementing the central assumption $\phi = 1$ of the Romer (1990) model, the growth rate of the economy can be written as

$$g = \frac{\dot{A}}{A} = \lambda L - \frac{r - \delta}{\alpha} \tag{23}$$

From equation (5) and via the fact that $\dot{C}/C = g$, we arrive at the following expression for the interest rate

$$r = g\sigma + \rho + \delta \tag{24}$$

which can be inserted into equation (23) to obtain the equilibrium growth rate

$$g = \frac{\lambda L \alpha - \rho}{\alpha + \sigma}. \tag{25}$$

This equilibrium growth rate is equivalent to the steady state growth rate $\bar{g}$ because the right hand side of equation (25) is constant due to the fact that $\dot{L}/L = 0$. There are two remarkable consequences of this expression for the steady state growth rate: first of all, a scale effect appears in the sense that a larger population size leads to faster economic growth; secondly, increases in the parameters that reduce employment of scientists lead to decreases in the long-run economic growth rate (see also Romer (1990)).

## 2.5 The Jones (1995) case

After implementing the central assumption $\phi < 1$ of the Jones (1995) model, the growth rate of the economy can be written as

$$g = \frac{\dot{A}}{A} = \frac{\lambda L}{A^{1-\phi}} - \frac{(r - \delta)}{\alpha} \tag{26}$$

and we can again insert equation (24) to solve for the equilibrium growth rate

$$g = \frac{\lambda L \alpha - \rho A^{1-\phi}}{A^{1-\phi}(\alpha + \sigma)}. \tag{27}$$

Due to $\dot{L}/L > 0$ and the fact that $A$ shows up on the right hand side of this equation, it is not yet constant, i.e. the equilibrium growth rate is not equivalent to the steady state growth rate of the economy. In Appendix A we derive the steady state growth rate as

$$\bar{g} = \frac{n}{1 - \phi} \tag{28}$$

which increases in the strength of technology spillovers, $\phi$, does not depend on the population size, $L$, but increases in the rate of population growth, $n$. If the population size is constant, long-run per capita output growth eventually ceases to exist (see Jones (1995)).

# 3 Introducing population ageing

In this section we introduce population ageing in the spirit of Blanchard (1985) to the Romer (1990) case, since there the population size has to stay constant, and in the spirit of Buiter (1988) to the Jones (1995) case, since there the population size has to grow. First of all we assume that the total population of an economy consists of different cohorts that are distinguishable by their date of birth denoted as $t_0$. Each cohort consist of a measure $N(t_0, t)$ of individuals at a certain point in time $t > t_0$. In addition, we assume that individuals have to face a constant risk of death at each instant which we denote as $\mu$. Due to the law of large numbers this rate is equal to the fraction of individuals dying at each instant. In the Romer (1990) case the population does not grow and therefore the birth rate is also equal to $\mu$, whereas in the Jones (1995) case the population grows at rate $n = \beta - \mu$, where $\beta > \mu$ is the birth rate.

## 3.1 Consumption sector

The discounted stream of an individual's lifetime utility can be written as

$$U = \int_{t_0}^{\infty} e^{-(\rho+\mu)(\tau - t_0)} \left( \frac{c^{1-\sigma} - 1}{1 - \sigma} \right) d\tau, \tag{29}$$

where $t_0$ refers to the date of birth of a certain cohort and the mortality rate $\mu$ augments the pure discount rate of individuals. The reason is that people face the risk of death and therefore do not like to postpone consumption to the same extent as in case of no lifetime uncertainty. We implement the assumption of Yaari (1965) that individuals insure themselves against the risk of dying with positive assets by using their whole savings to buy actuarial notes of a fair life-insurance company. This company redistributes wealth of individuals who died to those who survived within a certain cohort and therefore the real rate of return is augmented by the mortality rate. Consequently, the modified wealth constraint of individuals reads

$$\dot{k} = (r + \mu - \delta)k + wl - c. \tag{30}$$

Again we take the normalization $l \equiv 1$. In this case the individual Euler equations are shown in Appendix A to equal

$$\dot{c} = \frac{(r - \rho - \delta)}{\sigma} c \tag{31}$$

in case of a stationary population and

$$\dot{c} = \frac{(r - \rho - n - \delta)}{\sigma} c \tag{32}$$

in case of a growing population. They are similar to the Euler equations without lifetime uncertainty. However, our economy does not feature one single representative individual in this setting and we have to use certain aggregation rules to come up with aggregate Euler equations and aggregate laws of motion for capital.

## 3.2 Aggregation in case of a constant population

In the modified case, agents are heterogeneous with respect to age and therefore also with respect to accumulated wealth because older agents have had more time to build up positive assets. In order to get to the law of motion for aggregate capital and to the economy-wide Euler equation, we have to apply the following rules to aggregate over all cohorts alive at time $t$ (see also Heijdra and van der Ploeg (2002)):

$$K(t) \equiv \int_{-\infty}^{t} k(t_0, t) N(t_0, t) dt_0, \tag{33}$$

$$C(t) \equiv \int_{-\infty}^{t} c(t_0, t) N(t_0, t) dt_0. \tag{34}$$

By applying our demographic assumptions we can rewrite this as

$$C(t) \equiv \mu N \int_{-\infty}^{t} c(t_0, t) e^{\mu(t_0 - t)} dt_0 \tag{35}$$

$$K(t) \equiv \mu N \int_{-\infty}^{t} k(t_0, t) e^{\mu(t_0 - t)} dt_0 \tag{36}$$

because in case of a stationary population, each cohort is of size $\mu N e^{\mu(t_0-t)}$ at a certain point in time $t > t_0$[4]. After carrying out the calculations described in Appendix A, we arrive at the following expressions for the law of motion of aggregate capital and for the aggregate Euler equation

$$\dot{K} = (r - \delta)K(t) - C(t) + W(t), \tag{37}$$

$$\frac{\dot{C}(t)}{C(t)} = \frac{(r - \rho - \delta)}{\sigma} - \mu\Omega, \tag{38}$$

where we denote $\frac{C(t)-C(t,t)}{C(t)}$ as $\Omega$. Due to the fact that aggregate consumption, $C(t)$, is always higher than consumption of the newborns, $C(t,t)$, it holds that $\Omega \in [0, 1]$. Therefore aggregate consumption growth will always be lower than individual consumption growth. The reason is that at each instant, a fraction $\mu$ of older and therefore wealthier individuals die and they are replaced by poorer newborns. Since the latter can afford less consumption than the former, the turnover of generations slows down aggregate consumption growth (see also Heijdra and van der Ploeg (2002)).

## 3.3 Aggregation in case of a growing population

In case of the Jones (1995) model, population growth is allowed for. The aggregation rules in such a setting remain the same as in the previous subsection but the demographic assumptions change because the rate of birth $\beta$ has to exceed the mortality rate $\mu$. Therefore the population grows at rate $n = \beta - \mu$ and we normalize the initial population size to $L(0)$ such that we can write the size of a cohort born at $t_0 < t$ at a certain point in time as (see Appendix A):

$$N(t_0, t) = \beta L(0) e^{\beta t_0} e^{-\mu t}. \tag{39}$$

[4]Consequently, we have that $\int_{-\infty}^{t} \mu N e^{\mu(t_0-t)} dt_0 = L$ holds for the total population size at time $t$.

Integrating over all cohorts alive yields the population size as

$$L(t) = \beta L(0)e^{-\mu t}\int_{-\infty}^{t} e^{\beta t_0}dt_0. \tag{40}$$

Therefore we can define the aggregate capital stock and aggregate consumption according to

$$C(t) \equiv \beta L(0)e^{-\mu t}\int_{-\infty}^{t} c(t_0,t)e^{\beta t_0}dt_0 \tag{41}$$

$$K(t) \equiv \beta L(0)e^{-\mu t}\int_{-\infty}^{t} k(t_0,t)e^{\beta t_0}dt_0. \tag{42}$$

After carrying out the calculations described in Appendix A, we arrive at the aggregate law of motion for capital and the aggregate Euler equation

$$\dot{K} = (r+\mu-\beta-\delta)K(t) - C(t) + W(t), \tag{43}$$

$$\frac{\dot{C}(t)}{C(t)} = \frac{(r-\rho-\beta+\mu-\delta)}{\sigma} - \beta\Omega, \tag{44}$$

where we again denote $\frac{C(t)-C(t,t)}{C(t)}$ as $\Omega$. Note that the aggregate Euler equation is the same as in case of a constant population size, such that again economy-wide consumption growth falls short of individual consumption growth. Furthermore, we can state the following Lemma that holds in the Romer (1990) case as well as in the Jones (1995) case:

**Lemma 1.** *The term $\Omega$ is constant over time.*

*Proof.* Due to the fact that $\Omega$ can be expressed as (see Appendix A):

$$\Omega = (\rho+\mu)\frac{F(t)}{C(t)} \tag{45}$$

we see that it is constant as long as aggregate financial wealth, $F(t)$, and aggregate consumption, $C(t)$, grow at the same rate. Since there are no transitional dynamics because $\dot{p}_A = 0$, the aggregate capital stock immediately jumps to its optimal steady state value. Consequently, the economy never finds itself on a transition path, where capital accumulates faster than consumption grows. □

## 3.4 The steady state growth rate in the Romer (1990) case

To calculate steady state growth rate in the Romer (1990) case, we use the aggregate Euler equation for a constant population size to get the following expression for the interest rate

$$r \;=\; (g+\mu\Omega)\sigma+\rho+\delta \tag{46}$$

which we insert into equation (23) such that the equilibrium growth rate becomes

$$g \equiv \;=\; \frac{\lambda L\alpha-\rho-\mu\Omega\sigma}{\alpha+\sigma}. \tag{47}$$

The equilibrium growth rate is again equivalent to the steady state growth rate $\bar{g}$ because the right hand side of equation (47) is constant. At this stage, we can state the first central result:

**Proposition 1.** *In case of endogenous growth in the spirit of Romer (1990), increasing longevity has a positive effect on the steady state growth rate of an economy.*

*Proof.* The derivative of equation (47) with respect to mortality is equal to

$$\frac{\partial\bar{g}}{\partial\mu}=-\frac{\Omega\sigma}{\alpha+\sigma}$$

which is unambiguously negative because $\Omega$ and $\sigma$ are positive and $\alpha\in[0,1]$. As an increase in longevity is represented by a decrease in mortality $\mu$, the proposition holds. □

The intuition for this finding is that the planning horizon of individuals expands with longevity. Consequently, investments into new technologies have longer time horizons to pay off. This leads individuals to allocate more of their income to investments into technologies and less of their income to current consumption. Due to the growth effect of this shift, they are even overcompensated for the initial sacrifice by increases in lifetime consumption.

## 3.5 The steady state growth rate in the Jones (1995) case

In Appendix A we show that the equilibrium growth rate of the economy in the Jones (1995) case is equal to

$$g=\frac{\lambda L\alpha-(\rho+\beta-\mu+\beta\Omega\sigma)A^{1-\phi}}{A^{1-\phi}(\alpha+\sigma)}. \tag{48}$$

Note again that this is not yet the steady state growth rate because the right hand side of the equation is not constant. Therefore we can state the second central result:

**Proposition 2.** *In case of semi-endogenous growth in the spirit of Jones (1995), increasing longevity has a positive effect on the equilibrium growth rate of an economy during the transition period.*

*Proof.* Plugging equation (40) into equation (48) and taking the derivative with respect to mortality yields

$$\frac{\partial g}{\partial \mu} = -\frac{\alpha\beta\lambda\mu L(0)e^{-\mu t}\int_{-\infty}^{t} e^{\beta t_0}dt_0 + A^{1-\phi}}{A^{1-\phi}(\alpha+\sigma)}$$

which is unambiguously negative because $\mu$, $\lambda$, $\beta$, $\sigma$, $\alpha$, $L(0)$, $\Omega$ and $A$ are positive. As an increase in longevity is represented by a decrease in mortality $\mu$, the proposition holds. □

There are two reasons for this finding: First, the same force as compared to the Romer (1990) case works in the sense that the planning horizon of individuals expands and therefore investments into new technologies increase. There is, however, an additional effect because population growth accelerates if mortality decreases and fertility stays constant. Consequently, the flow of scientists into the R & D sector accelerates as well and a higher growth rate in the number of blueprints can be sustained.

However, the right hand side of equation (48) is not constant, so it does not yet represent the steady state growth rate in the Jones (1995) case. We search for an expression where the growth rate is constant and carry out the associated calculations in Appendix A. This leads us to the expression

$$\bar{g} = \frac{\beta - \mu}{1 - \phi} \tag{49}$$

for the steady state growth rate of the economy and therefore we state the third central result:

**Proposition 3.** *In case of semi-endogenous growth in the spirit of Jones (1995), increasing longevity raises the steady state growth rate of an economy.*

*Proof.* The derivative of equation (49) with respect to mortality is equal to

$$\frac{\partial \bar{g}}{\partial \mu} = -\frac{1}{1-\phi}$$

which is unambiguously negative because $\phi < 1$ is the central assumption in the Jones (1995) case. As an increase in longevity is represented by a decrease in mortality $\mu$, the proposition holds. □

The interpretation for this finding is that an increase in the population growth rate represents a permanent increase in the flow of resources devoted to R & D and therefore a higher growth rate of the number of patents can be sustained. In contrast, an expansion of the planning horizon of individuals and the associated increase in investments into new technologies represents a level effect only. The resources devoted to R & D increase once and for all which is sufficient to speed up the growth rate of the number of patents in the medium-run but insufficient to sustain this increase over longer time horizons.

# 4 Conclusions

We set up a model for endogenous technological change that nests the Romer (1990) and the Jones (1995) frameworks. Afterwards we generalize this model class by introducing finite individual planning horizons and thereby allowing for overlapping generations and heterogeneous individuals. As compared to the standard case of zero mortality and infinite planning horizons, we show that the steady state growth rates in both settings are lower when mortality is present. The explanation for this result is that individuals have shorter planning horizons and therefore they are not willing to invest in R & D to the same extent as in case of zero mortality. The reason is that revenues of R & D largely accrue in the future and people who face the risk of death discount the future more heavily than infinitely lived individuals.

Altogether our framework allows us to study the effects of increases in longevity on the long-run economic growth perspectives of a certain economy. In case of the Romer (1990) model, increasing longevity is not associated with population growth. Instead, the mean age of the population increases, which positively affects the per capita growth rate in the steady state. In case of the Jones (1995) model, increasing longevity not only raises the mean age of a society, but also increases the population growth rate. Consequently, there are positive effects of increases in mortality on the equilibrium growth rate during the transition period as well as on the steady state growth rate in the long run.

From an applied perspective, the conclusion of our model is that population ageing does not itself hamper technological progress and therefore economic prosperity. Instead, it might be associated with increasing private investments into knowledge creation as the individual time horizon expands such that these investments are more likely to pay off. This effect is also supported by empirical evidence which finds that an increasing life expectancy has a positive influence on per capita output growth (see for example Kelley and Schmidt (2005)). Of course there might exist other – sometimes negative – effects of ageing on per capita growth in the medium run (e.g. problems in financing pensions or decreases in aggregate productivity) from which we explicitly abstracted by concentrating on the evolution of technology in the long run. However, regarding these issues, extensive research has been carried out recently (see for example Bloom et al. (2008)).

Finally, we can state that there is scope for further research because a constant mortality rate is still at odds with reality and one could try to introduce age dependent mortality rates. Another promising field for additional investigations could be to introduce heterogeneity of researchers with respect to age. These issues are on top of our research agenda.


## Acknowledgements

I would like to thank Dalkhat Ediev (Vienna Institute of Demography), Theresa Grafeneder-Weissteiner (Vienna University of Economics and Business), Ingrid Kubin (Vienna University of Economics and Business), Alexia Prskawetz (Vienna University of Technology), Holger Strulik (University of Hannover), Vladimir Veliov (Vienna University of Technology) and Stefan Wrzaczek (Vienna University of Technology) for useful comments and suggestions. Financial support by the Vienna Science and Technology Fund (WWTF) in its "Mathematics and..." call 2007 is greatly acknowledged.

# A Derivations

**The individual Euler equation without ageing:** The current value Hamiltonian is

$$H = \left(\frac{c^{1-\sigma}-1}{1-\sigma}\right) + \lambda\left[(r-\delta)k + w - c\right]$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial H}{\partial c} &= c^{-\sigma} - \lambda \overset{!}{=} 0 \\
\Rightarrow c^{-\sigma} &= \lambda \qquad (50) \\
\frac{\partial H}{\partial k} &= (r-\delta)\lambda \overset{!}{=} \rho\lambda - \dot{\lambda} \\
\Rightarrow \dot{\lambda} &= (\rho + \delta - r)\lambda. \qquad (51)
\end{aligned}$$

Taking the time derivative of equation (50)

$$-\sigma c^{-\sigma-1}\dot{c} = \dot{\lambda}$$

and plugging it into equation (51) yields

$$\begin{aligned}
-\sigma c^{-\sigma-1}\dot{c} &= (\rho + \delta - r)\lambda \\
c^{-\sigma-1}\dot{c} &= \frac{(r-\rho-\delta)c^{-\sigma}}{\sigma} \\
\frac{\dot{c}}{c} &= \frac{(r-\rho-\delta)}{\sigma}
\end{aligned}$$

which is the standard Euler equation. In case of the Jones (1995) model with population growth, the discount rate has to be augmented by the population growth rate (see Acemoglu (2009)) such that the individual Euler equation without ageing becomes

$$\frac{\dot{c}}{c} = \frac{(r-\rho-n-\delta)}{\sigma}.$$

**Operating profits for intermediate goods producers:** Profits of intermediate goods producers can be obtained via equation (10) as

$$\begin{aligned}
\pi &= \frac{r}{\alpha}x - rx \\
&= \frac{r-\alpha r}{\alpha}x \\
&= \frac{(1-\alpha)}{\alpha}rx \\
&= (1-\alpha)px \\
&= (1-\alpha)\alpha L_y^{1-\alpha}k_i^{\alpha} \\
&= (1-\alpha)\alpha\frac{Y}{A}
\end{aligned}$$

**Labour input in both sectors:** We determine the fraction of workers employed in the final goods sector and in the R & D sector by using equation (16):

$$\begin{aligned}
p^A\lambda A^{\phi} &= (1-\alpha)\frac{Y}{L_y} \\
\frac{(1-\alpha)\alpha Y}{(r-\delta)A}\lambda A^{\phi} &= (1-\alpha)\frac{Y}{L_y} \\
\frac{(1-\alpha)\alpha}{(r-\delta)A^{1-\phi}}\lambda &= (1-\alpha)\frac{1}{L_y} \\
L_y &= \frac{(r-\delta)A^{1-\phi}}{\alpha\lambda} \\
\Rightarrow L_A &= L - \frac{(r-\delta)A^{1-\phi}}{\alpha\lambda},
\end{aligned}$$

where the last line follows from labour market clearing, i.e. $L = L_A + L_Y$.

**Steady state growth rate in the Romer (1990) case:** We insert equation (24) into equation (23) to get

$$\begin{aligned}
g &= \lambda L - \frac{g\sigma+\rho}{\alpha} \\
&= \lambda L - \frac{g\sigma}{\alpha} - \frac{\rho}{\alpha} \\
\Rightarrow g\left(1+\frac{\sigma}{\alpha}\right) &= \lambda L - \frac{\rho}{\alpha} \\
g\left(\frac{\alpha^2+(1-\alpha)\sigma}{\alpha^2}\right) &= \frac{\lambda L\alpha^2-(1-\alpha)\rho}{\alpha^2} \\
g \equiv \bar{g} &= \frac{\lambda L\alpha-\rho}{\alpha+\sigma},
\end{aligned}$$

where $\bar{g}$ denotes the steady state growth rate.

**Derivation of the steady state growth rate in the Jones (1995) case:** We insert equation (24) into equation (26) to get

$$\begin{aligned} g &= \frac{\lambda L}{A^{1-\phi}} - \frac{(g\sigma + \rho)}{\alpha} \\ &= \frac{\lambda L}{A^{1-\phi}} - \frac{g\sigma}{\alpha} - \frac{\rho}{\alpha} \\ \Rightarrow g\left(1 + \frac{\sigma}{\alpha}\right) &= \frac{\lambda L}{A^{1-\phi}} - \frac{\rho}{\alpha} \\ g\left(\frac{\alpha + \sigma}{\alpha}\right) &= \frac{\lambda L\alpha - \rho A^{1-\phi}}{A^{1-\phi}\alpha} \\ g &= \frac{\lambda L\alpha - \rho A^{1-\phi}}{A^{1-\phi}(\alpha + \sigma)}. \end{aligned}$$

Since the right hand side is not constant, this is not yet the steady state growth rate of the economy. We search for an expression where the growth rate is constant, i.e. the growth rate of the growth rate is zero. Therefore we separate the expression to obtain

$$g = \frac{\lambda L\alpha}{A^{1-\phi}(\alpha + \sigma)} - \frac{\rho}{\alpha + \sigma}.$$

Taking the time derivative yields

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\lambda \dot{L}\alpha A^{1-\phi}(\alpha + \sigma) - \lambda L\alpha(1-\phi)A^{-\phi}\dot{A}(\alpha + \sigma)}{\left[A^{1-\phi}(\alpha + \sigma)\right]^2} \\ &= \frac{\lambda \dot{L}\alpha - \lambda L\alpha(1-\phi)g}{A^{1-\phi}(\alpha + \sigma)} \end{aligned} \tag{52}$$

In the steady state, the left hand side is equal to zero such that we can obtain the steady state growth rate as

$$\bar{g} = \frac{n}{1-\phi}.$$

**The individual Euler equation with ageing:** The current value Hamiltonian is

$$H = \left(\frac{c^{1-\sigma} - 1}{1-\sigma}\right) + \lambda\left[(r + \mu - \delta)k + w - c\right].$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial H}{\partial c} &= c^{-\sigma} - \lambda \overset{!}{=} 0 \\
\Rightarrow c^{-\sigma} &= \lambda \qquad (53)\\
\frac{\partial H}{\partial k} &= (r + \mu - \delta)\lambda \overset{!}{=} (\rho + \mu)\lambda - \dot{\lambda} \\
\Rightarrow \dot{\lambda} &= (\rho + \delta - r)\lambda. \qquad (54)
\end{aligned}
$$

Taking the time derivative of equation (53)

$$
-\sigma c^{-\sigma-1}\dot{c} = \dot{\lambda}
$$

and plugging it into equation (54) yields

$$
\begin{aligned}
-\sigma c^{-\sigma-1}\dot{c} &= (\rho + \delta - r)\lambda \\
c^{-\sigma-1}\dot{c} &= \frac{(r - \rho - \delta)c^{-\sigma}}{\sigma} \\
\frac{\dot{c}}{c} &= \frac{(r - \rho - \delta)}{\sigma}
\end{aligned}
$$

which is the individual Euler equation. In case of the Jones (1995) model with population growth, the discount rate has to be augmented by the population growth rate (see Acemoglu (2009)) such that the individual Euler equation with ageing becomes

$$
\frac{\dot{c}}{c} = \frac{(r - \rho - n - \delta)}{\sigma}.
$$

**Aggregate capital and aggregate consumption in the Romer (1990) case:** Differentiating equations (35) and (36) with respect to time yields

$$
\begin{aligned}
\dot{C}(t) &= \mu N \left[ \int_{-\infty}^{t} \dot{c}(t_0, t) e^{\mu(t_0 - t)} dt_0 - \mu \int_{-\infty}^{t} c(t_0, t) e^{\mu(t_0 - t)} dt_0 \right] + \mu N c(t, t) - 0 \\
&= \mu N c(t, t) - \mu C(t) + \mu N \int_{-\infty}^{t} \dot{c}(t_0, t) e^{-\mu(t - t_0)} dt_0 \qquad (55)\\
\dot{K}(t) &= \mu N \left[ \int_{-\infty}^{t} \dot{k}(t_0, t) e^{\mu(t_0 - t)} dt_0 - \mu \int_{-\infty}^{t} k(t_0, t) e^{\mu(t_0 - t)} dt_0 \right] + \mu N k(t, t) - 0 \\
&= \mu N \underbrace{k(t, t)}_{=0} - \mu K(t) + \mu N \int_{-\infty}^{t} \dot{k}(t_0, t) e^{-\mu(t - t_0)} dt_0. \qquad (56)
\end{aligned}
$$

From equation (30) it follows that

$$\begin{aligned}
\dot{K}(t) &= -\mu K(t) + \mu N \int_{-\infty}^{t} \left[ (r+\mu-\delta)k(t_0,t) + w(t) - c(t_0,t) \right] e^{-\mu(t-t_0)} dt_0 \\
&= -\mu K(t) + (r+\mu-\delta)\mu N \int_{-\infty}^{t} k(t_0,t) e^{-\mu(t-t_0)} dt_0 \\
&\quad -\mu N \int_{-\infty}^{t} c(t_0,t) e^{-\mu(t-t_0)} dt_0 + N \left( \frac{\mu w(t) e^{-\mu(t-t_0)}}{\mu} \right)_{-\infty}^{t} \\
&= -\mu K(t) + (r+\mu-\delta)K(t) - C(t) + W(t) \\
&= (r-\delta)K(t) - C(t) + W(t)
\end{aligned}$$

which is the aggregate law of motion for capital. Reformulating an agent's optimization problem subject to its lifetime budget restriction, stating that the present value of lifetime consumption expenditures have to be equal to the present value of lifetime wage income plus initial assets, yields the optimization problem

$$\begin{aligned}
\max_{c(t_0,\tau)} \quad U &= \int_{t}^{\infty} e^{(\rho+\mu)(t-\tau)} \left( \frac{c(t_0,\tau)^{1-\sigma} - 1}{1-\sigma} \right) d\tau \\
s.t. \qquad k(t_0,t) &+ \int_{t}^{\infty} w(\tau) e^{-R^A(t,\tau)} d\tau = \int_{t}^{\infty} c(t_0,\tau) e^{-R^A(t,\tau)} d\tau,
\end{aligned} \tag{57}$$

where $R^A(\tau,t) = \int_t^\tau (r(s)+\mu-\delta)ds$. The FOC to this optimization problem is

$$c(t_0,\tau)^{-\sigma} e^{(\rho+\mu)(t-\tau)} = \lambda(t) e^{-R^A(t,\tau)}.$$

In period $(\tau = t)$ we have

$$c(t_0,\tau) = \frac{1}{\lambda^{1/\sigma}}.$$

Therefore we can write

$$\begin{aligned}
c(t_0,\tau)^{-\sigma} e^{(\rho+\mu)(t-\tau)} &= c(t_0,\tau)^{-\sigma} e^{-R^A(t,\tau)} \\
c(t_0,\tau) e^{(\rho+\mu)(t-\tau)} &= c(t_0,\tau) e^{-R^A(t,\tau)}.
\end{aligned}$$

Integrating and using equation (57) yields

$$
\begin{aligned}
\int_t^\infty c(t_0,\tau)e^{(\rho+\mu)(t-\tau)}d\tau &= \int_t^\infty c(t_0,\tau)e^{-R^A(t,\tau)}d\tau \\
\frac{c(t_0,\tau)}{\rho+\mu}\left[-e^{(\rho+\mu)(t-\tau)}\right]_t^\infty &= \underbrace{k(t_0,t)}_{f(t_0,t)} + \underbrace{\int_t^\infty w(\tau)e^{-R^A(t,\tau)}d\tau}_{h(t)} \\
\Rightarrow c(t_0,\tau) &= (\rho+\mu)\left[f(t_0,t)+h(t)\right], \qquad (58)
\end{aligned}
$$

where $f$ refers to financial wealth and $h$ to human wealth of individuals. The latter does not depend on the date of birth because productivity is age independent. The last line holds because $a^0 = 1$ for any $a$. Therefore optimal consumption in the planning period is proportional to total wealth with a marginal propensity to consume of $\rho + \mu$. Aggregate consumption evolves according to

$$
\begin{aligned}
C(t) &\equiv \mu N \int_{-\infty}^t c(t_0,t)e^{\mu(t_0-t)}dt_0 \\
&= \mu N \int_{-\infty}^t e^{\mu(t_0-t)}(\rho+\mu)\left[f(t_0,t)+h(t)\right]dt_0 \\
&= (\rho+\mu)F(t) + \mu N(\rho+\mu)\int_{-\infty}^t e^{\mu(t_0-t)}h(t)dt_0 \\
&= (\rho+\mu)\left[F(t)+H(t)\right]. \qquad (59)
\end{aligned}
$$

Note that newborns do not have financial wealth because there are no bequests. Therefore

$$
\begin{aligned}
c(t,t) &= (\rho+\mu)h(t) \\
C(t,t) &= (\rho+\mu)H(t) \qquad (60)
\end{aligned}
$$

holds for each newborn individual and each newborn cohort, respectively. Putting equations (55), (31), (59) and (60) together yields

$$\begin{aligned}
\dot{C}(t) &= \mu N c(t,t) - \mu C(t) + \mu N \int_{-\infty}^{t} \dot{c}(t_0,t) e^{-\mu(t-t_0)} dt_0 \\
&= \mu N(\rho+\mu) h(t) - \mu(\rho+\mu)\left[F(t)+H(t)\right] \\
&+ \mu N \int_{-\infty}^{t} \dot{c}(t_0,t) e^{-\mu(t-t_0)} dt_0 \\
&= \mu(\rho+\mu) H(t) - \mu(\rho+\mu)\left[F(t)+H(t)\right] + \\
&\quad \mu N \int_{-\infty}^{t} \frac{(r-\rho-\delta)}{\sigma} c(t_0,t) e^{-\mu(t-t_0)} dt_0 \\
&= \mu(\rho+\mu) H(t) - \mu(\rho+\mu)\left[F(t)+H(t)\right] + \frac{(r-\rho-\delta)}{\sigma} C(t) \\
\Rightarrow \frac{\dot{C}(t)}{C(t)} &= \frac{(r-\rho-\delta)}{\sigma} + \frac{\mu(\rho+\mu) H(t) - \mu(\rho+\mu)\left[F(t)+H(t)\right]}{C(t)} \\
&= \frac{(r-\rho-\delta)}{\sigma} - \mu(\rho+\mu)\frac{F(t)}{C(t)} \\
&= \frac{(r-\rho-\delta)}{\sigma} - \mu \underbrace{\frac{C(t)-C(t,t)}{C(t)}}_{\in(0,1)}
\end{aligned}$$

which is the aggregate Euler equation that differs from the individual Euler equation by the term $-\mu\frac{C(t)-C(t,t)}{C(t)}$.

**Aggregate capital and aggregate consumption in the Jones (1995) case:** Using our demographic assumptions we can write the size of a cohort born at $t_0 < t$ at time $t$ as

$$\begin{aligned}
N(t_0,t) &= \beta L(t_0) e^{-\mu(t-t_0)} \\
&= \beta L(0) e^{n t_0} e^{-\mu(t-t_0)} \\
&= \beta L(0) e^{\beta t_0 - \mu t_0} e^{-\mu t + \mu t_0} \\
&= \beta L(0) e^{\beta t_0} e^{-\mu t}.
\end{aligned}$$

Integrating over all cohorts yields the population size as

$$\begin{aligned}
L(t) &= \int_{-\infty}^{t} \beta L(0) e^{\beta t_0} e^{-\mu t} dt_0 \\
&= \beta L(0) e^{-\mu t} \int_{-\infty}^{t} e^{\beta t_0} dt_0.
\end{aligned}$$

Differentiating equations (42) and (41) with respect to time yields:

$$
\begin{aligned}
\dot{C}(t) &= \beta L(0)e^{-\mu t}\left[\int_{-\infty}^{t} \dot{c}(t_0,t)e^{\beta(t_0)}dt_0 - \beta\int_{-\infty}^{t} c(t_0,t)e^{\beta t_0}dt_0\right] \\
&+ \beta L(0)e^{-\mu t}c(t,t)e^{\beta t} - 0 \\
&= \beta L(0)e^{-\mu t}c(t,t)e^{\beta t} - \beta C(t) + \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \dot{c}(t_0,t)e^{\beta t_0}dt_0
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\dot{K}(t) &= \beta L(0)e^{-\mu t}\left[\int_{-\infty}^{t} \dot{k}(t_0,t)e^{\beta(t_0)}dt_0 - \beta\int_{-\infty}^{t} k(t_0,t)e^{\beta t_0}dt_0\right] \\
&+ \beta L(0)e^{-\mu t}k(t,t)e^{\beta t} - 0 \\
&= \beta L(0)e^{-\mu t}\underbrace{k(t,t)}_{=0} - \beta K(t) + \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \dot{k}(t_0,t)e^{\beta t_0}dt_0.
\end{aligned} \tag{62}
$$

From equation (30) it follows that

$$
\begin{aligned}
\dot{K}(t) &= -\beta K(t) + \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \left[(r+\mu-\delta)k(t_0,t) + w(t) - c(t_0,t)\right]e^{\beta t_0}dt_0 \\
&= -\beta K(t) + (r+\mu-\delta)\beta L(0)e^{-\mu t}\int_{-\infty}^{t} k(t_0,t)e^{\beta t_0}dt_0 \\
&\quad -\beta L(0)e^{-\mu t}\int_{-\infty}^{t} c(t_0,t)e^{\beta t_0}dt_0 + L(0)e^{-\mu t}\left(\frac{\beta w(t)e^{\beta t_0}}{\beta}\right)_{-\infty}^{t} \\
&= -\beta K(t) + (r+\mu-\delta)K(t) - C(t) + W(t) \\
&= (r+\mu-\beta-\delta)K(t) - C(t) + W(t)
\end{aligned}
$$

which is the aggregate law of motion for capital. Note that the definition of aggregate wages is $W(t) = L(0)w(t)e^{\beta-\gamma}$. By making use of equation (58) we can write aggregate consumption as

$$
\begin{aligned}
C(t) &\equiv \beta L(0)e^{-\mu t}\int_{-\infty}^{t} c(t_0,t)e^{\beta t_0}dt_0 \\
&= \beta L(0)e^{-\mu t}\int_{-\infty}^{t} e^{\beta t_0}(\rho+\mu)\left[f(t_0,t) + h(t)\right]dt_0 \\
&= (\rho+\mu)F(t) + \beta L(0)e^{-\mu t}(\rho+\mu)\int_{-\infty}^{t} e^{\beta t_0}h(t)dt_0 \\
&= (\rho+\mu)\left[F(t) + H(t)\right].
\end{aligned} \tag{63}
$$

Note that the following definitions apply: $F(t) = \beta L(0)e^{-\mu t}\int_{-\infty}^{t} e^{\beta t_0}f(t,t_0)dt_0$ and $H(t) = L(0)e^{(\beta-\mu)t}h(t)$. Newborns do not have financial wealth because there are no bequests, therefore

$$
\begin{aligned}
c(t,t) &= (\rho+\mu)h(t) \\
C(t,t) &= (\rho+\mu)H(t) \qquad (64)
\end{aligned}
$$

holds for each newborn individual and each newborn cohort, respectively. Putting equations (61), (32), (63) and (64) together yields

$$
\begin{aligned}
\dot{C}(t) &= \beta L(0)e^{-\mu t}c(t,t)e^{\beta t} - \beta C(t) + \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \dot{c}(t_0,t)e^{\beta t_0}dt_0 \\
&= \beta L(0)e^{(\beta-\mu)t}(\rho+\mu)h(t) - \beta(\rho+\mu)\left[F(t)+H(t)\right] \\
&+ \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \dot{c}(t_0,t)e^{\beta t_0}dt_0 \\
&= \beta(\rho+\mu)H(t) - \beta(\rho+\mu)\left[F(t)+H(t)\right] + \\
&\quad \beta L(0)e^{-\mu t}\int_{-\infty}^{t} \frac{(r-\rho-\delta)}{\sigma}c(t_0,t)e^{\beta t_0}dt_0 \\
&= \beta(\rho+\mu)H(t) - \beta(\rho+\mu)\left[F(t)+H(t)\right] + \frac{(r-\rho-\delta)}{\sigma}C(t) \\
\Rightarrow \frac{\dot{C}(t)}{C(t)} &= \frac{(r-\rho-\beta+\mu-\delta)}{\sigma} + \frac{\beta(\rho+\mu)H(t) - \beta(\rho+\mu)\left[F(t)+H(t)\right]}{C(t)} \\
&= \frac{(r-\rho-\beta+\mu-\delta)}{\sigma} - \beta(\rho+\mu)\frac{F(t)}{C(t)} \\
&= \frac{(r-\rho-\beta+\mu-\delta)}{\sigma} - \beta\underbrace{\frac{C(t)-C(t,t)}{C(t)}}_{\in(0,1)}
\end{aligned}
$$

which is the aggregate Euler equation that differs from the individual Euler equation by the term $-\beta\frac{C(t)-C(t,t)}{C(t)}$.

**The steady state growth rate in the Romer (1990) case with an ageing population:** We insert equation (46) into equation (23) to solve for the equilibrium growth rate

$$\begin{aligned}
g &= \frac{\dot{A}}{A} = \lambda L - \frac{(g+\mu\Omega)\sigma + \rho}{\alpha} \\
&= \lambda L - \frac{g\sigma}{\alpha} - \frac{(\rho + \mu\Omega\sigma)}{\alpha} \\
\Rightarrow g\left(1 + \frac{\sigma}{\alpha}\right) &= \lambda L - \frac{(\rho + \mu\Omega\sigma)}{\alpha} \\
g\left(\frac{\alpha + \sigma}{\alpha}\right) &= \frac{\lambda L\alpha - \rho - \mu\Omega\sigma}{\alpha} \\
g \equiv \bar{g} &= \frac{\lambda L\alpha - \rho - \mu\Omega\sigma}{\alpha + \sigma},
\end{aligned}$$

where again the steady state growth rate is equivalent to the equilibrium growth rate because the right hand side is constant.

**The steady state growth rate in the Jones (1995) case with an ageing population:** The growth rate of the economy is

$$g = \frac{\dot{A}}{A} = \frac{\lambda L}{A^{1-\phi}} - \frac{r - \delta}{\alpha}.$$

Since we have

$$\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= \frac{(r - \rho - \beta + \mu - \delta)}{\sigma} - \beta\Omega, \\
\Rightarrow r &= (g + \beta\Omega)\sigma + \rho + \beta - \mu + \delta,
\end{aligned}$$

we can solve for the equilibrium growth rate via

$$\begin{aligned}
g &= \frac{\lambda L}{A^{1-\phi}} - \frac{(g+\beta\Omega)\sigma + \rho + \beta - \mu}{\alpha} \\
&= \frac{\lambda L}{A^{1-\phi}} - \frac{g\sigma}{\alpha} - \frac{\rho + \beta - \mu + \beta\Omega\sigma}{\alpha} \\
\Rightarrow g\left(1 + \frac{\sigma}{\alpha}\right) &= \frac{\lambda L}{A^{1-\phi}} - \frac{\rho + \beta - \mu + \beta\Omega\sigma}{\alpha} \\
g\left(\frac{\alpha + \sigma}{\alpha}\right) &= \frac{\lambda L\alpha - (\rho + \beta - \mu + \beta\Omega\sigma)A^{1-\phi}}{A^{1-\phi}\alpha} \\
g &= \frac{\lambda L\alpha - (\rho + \beta - \mu + \beta\Omega\sigma)A^{1-\phi}}{A^{1-\phi}(\alpha + \sigma)}.
\end{aligned}$$

Since the right hand side is not constant, this is not yet the steady state growth rate of the economy. We search for an expression where the growth rate is constant.

Therefore we separate the expression to obtain

$$g = \frac{\lambda L \alpha}{A^{1-\phi}(\alpha+\sigma)} - \frac{\rho+\beta-\mu+\beta\Omega\sigma}{\alpha+\sigma}. \tag{65}$$

Taking the time derivative yields

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\lambda \dot{L} \alpha A^{1-\phi}(\alpha+\sigma) - \lambda L \alpha (1-\phi) A^{-\phi} \dot{A}(\alpha+\sigma)}{\left[A^{1-\phi}(\alpha+\sigma)\right]^2} \\ &= \frac{\lambda \dot{L} \alpha - \lambda L \alpha (1-\phi) g}{A^{1-\phi}(\alpha+\sigma)} \end{aligned} \tag{66}$$

In the steady state, the left hand side is equal to zero such that we can obtain the steady state growth rate as

$$\bar{g} = \frac{\beta-\mu}{1-\phi}.$$

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