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ON THE TANGENTIAL CONE CONDITION FOR ELECTRICAL IMPEDANCE TOMOGRAPHY*

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Abstract. We state some sufficient criteria for the tangential cone conditions to hold for the electrical impedance tomography problem. The results are based on Löwner convexity of the forward operator. As a consequence, we show that for conductivities that satisfy various properties, such as Hölder source conditions, finite-dimensionality, or certain monotonicity criteria, the tangential cone condition is verified.

Key words. impedance tomography, tangential cone condition, Löwner convexity

AMS subject classifications. 47J06, 65F22

1. Introduction. The electrical impedance tomography (EIT) problem is a classical inverse problem where the aim is to extract information about the conductivity from boundary measurements of current-voltage pairs. Starting with the definition of the problem in the seminal paper of Calderón [4], it has been investigated in various directions and now serves as a paradigmatic instance of a parameter identification problem from boundary measurements.

The common mathematical formulation is to consider solutions of the boundary value problem on a Lipschitz domain Ω,

$$\operatorname{div}(\gamma\nabla u) = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega. \tag{1.1}$$

The data for the inverse problem are multiple or infinitely many pairs of Cauchy data $(f, \gamma(\partial/\partial n)u|_{\partial\Omega})$ on the boundary, and the interest is to recover the conductivity $\gamma(x)$ in the interior Ω. As is typical for such an identification problem with only boundary data, this leads under the usual circumstances to a nonlinear severely ill-posed problem; without strong restrictions on the conductivity, one can at best only expect conditional logarithmic stability [13]. Various classical uniqueness and stability results are collected, for example, in [2] or [13].

In the following, we assume that the unknown conductivity can be written as a perturbation $\delta\gamma$ of a known background, which we take without loss of generality as 1. Thus, we assume throughout that

$$\gamma(x) = 1 + \delta\gamma(x), \quad \underline{\alpha} \leq \gamma(x) \leq \overline{\alpha}, \quad \text{a.e. in } \Omega, \tag{1.2}$$

with positive constants $\underline{\alpha}$ and $\overline{\alpha}$ to ensure ellipticity and stability of the partial differential equation.

In order to state the problem, it is convenient to frame it into operator-theoretic language. The above-mentioned Cauchy data (in the case of complete data) are equivalent to knowledge of the *Dirichlet-to-Neumann* (DtN) operator. We denote by Λ_γ the DtN operator for (1.1), i.e., the mapping

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f &\to \gamma \frac{\partial}{\partial n} u\Big|_{\partial\Omega}, \end{aligned} \tag{1.3}$$

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where u is a solution to (1.1), and, as defined below, $H^{1/2}$ is the usual fractional Sobolev space with constants factored out. We furthermore introduce the parameter-to-data map

$$F(\gamma) := \Lambda_\gamma - \Lambda_1, \tag{1.4}$$

where Λ_γ is the DtN map for (1.1) and Λ_1 is that for $\gamma = 1$. Since we are mainly interested in perturbations of the constant conductivity, we have subtracted the (known) background influence of Λ_1 from the data. Solving the inverse problem is then equivalent to inverting F. We give a precise definition of the associated spaces X and Y in the mapping $F : X \to Y$ in the next section.

The main theme of this article concerns not the solution of this problem but the investigation of the nonlinearity of F. This is highly relevant when applying regularization methods, in particular, iterative ones. Indeed, the convergence theory of iterative regularization methods such as Landweber's method requires some restrictions that quantify the deviation of the problem from a linear one. In this work, we focus on the well-known tangential cone conditions and their variants (cf. [7, 16, 18]).

For a general inverse problem with a differentiable parameter-to-data map F between Hilbert spaces, the so-called strong tangential cone condition [7, 18] is satisfied if, with F' denoting the Fréchet derivative of F, there is an η, $1 > \eta > 0$, such that

$$\|F(\tilde{x}) - F(x) - F'[x](\tilde{x} - x)\| \leq \eta \|F(\tilde{x}) - F(x)\| \tag{stc}$$

holds for all $\tilde{x}$ and x in a neighborhood of some x_0. A weaker version, the weak tangential cone condition [18], holds if an η, $1 > \eta > 0$, exists such that

$$(F(\tilde{x}) - F(x) - F'[x](\tilde{x} - x), F(\tilde{x}) - F(x))_Y \leq \eta \|F(\tilde{x}) - F(x)\|^2. \tag{wtc}$$

Moreover, the weak tangential cone condition with $\eta = 1$ reads as

$$(F'[x](\tilde{x} - x), F(\tilde{x}) - F(x))_Y \geq 0, \tag{qcon}$$

which yields a weaker condition than (wtc) that has been proposed in [16]. Note that by the parallelogram identity, the inequalities (wtc) and (qcon) may be equivalently rewritten as

$$\begin{aligned} &\|F(\tilde{x}) - F(x) - F'[x](\tilde{x} - x)\|^2 \\ &\qquad \leq (2\eta - 1)\|F(\tilde{x}) - F(x)\|^2 + \|F'[x](\tilde{x} - x)\|^2, \end{aligned} \tag{1.5}$$

where $\eta = 1$ in the case of (qcon).

These inequalities are central to the convergence theory of the nonlinear Landweber method and many other iterative regularization methods. They constitute a replacement for coercivity estimates, which cannot exist in the ill-posed case. It is a classical result that (under some standard additional assumptions) the strong tangential cone condition with $\eta \leq \frac{1}{2}$ implies strong convergence of the Landweber method [7, 14]. Similarly, the weak tangential cone condition [18] implies the nonexpansivity of the iteration, and, in particular, weak (subsequential) convergence of the iterates. (Of course, all this is in connection with appropriate stopping rules.) The condition (qcon) implies that the iterates stay in a neighborhood of the solution, which also yields weak (subsequential) convergence [16].

It is surprising that, in view of its importance, the tangential cone conditions for the impedance tomography problem could only be verified in a few special cases. For instance, Lechleiter and Rieder [17] have proven (stc) in a semidiscrete case, essentially by using a stability result for the discrete problem. Moreover, de Hoop, Qiu, and Scherzer [5] have

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verified (stc) for a class of piecewise constant conductivities being constant on finitely many regions. The proof is based on a Lipschitz stability result of Alessandrini and Vessella [1]. In both cases, the stability constants might become quite large; thus, the cone conditions can in practice only be theoretically verified in a very narrow neighborhood of the true solution.

Except for these few cases, the validity of the above tangential cone conditions is completely open, which is quite puzzling given the fact that the Landweber method has successfully been applied to the impedance tomography problem in many situations. Our article aims to gain further understanding of this fact (though without completely resolving it) by analyzing and establishing sufficient conditions for the tangential cone conditions. The main contribution is that a condition of the form

$$\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)^2\| \leq C\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)\|$$

suffices; see (4.3). This is established by making use of Löwner convexity and related estimates for the EIT problem.

REMARK 1.1. It is important to note that, to prove the convergence of the Landweber method, the above-mentioned tangential cone conditions do not have to hold for all elements in a neighborhood, but only for $\tilde{x}$ being the iterates of the method and x the exact solution. In this sense, we *do not* aim to prove (stc) or (wtc) for all $\tilde{x}$ and x but only for specific ones that satisfy, for instance, certain monotonicity conditions.

One result that is probably most relevant in practice is that the tangential cone conditions are satisfied for conductivities that satisfy certain monotonicity properties (e.g., a purely positive perturbation of the background conductivity). Further results concern a verification of (stc) for elements in a finite-dimensional space of if a Hölder source condition holds.

The article is organized as follows. In the next section we define the precise setup and provide some mathematical background on the Löwner ordering. In Section 3, we state some known Löwner convexity results together with a new operator-theoretic proof that allows us to obtain new, slightly improved inequalities. These results are used in Section 4 to state new sufficient conditions for the tangential cone conditions. We conclude with a discussion in Section 5.

2. Problem setup and operator estimates. We formulate some standard assumptions and specify the notation. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. We make use of the space of L^2-vector fields:

$$L^2(\Omega)^n := \left\{ v : \Omega \to \mathbb{R}^n \;\middle|\; \int_\Omega |v(x)|^2 < \infty \right\}.$$

The Dirichlet data f are canonically chosen in $\mathcal{H}^{1/2}$, which is the usual fractional Sobolev space with index $\frac{1}{2}$, or, alternatively, the space of all traces of $H^1(\Omega)$-functions. Since constant functions f are in the nullspace of Λ_γ, in the sequel we only consider the associated subspace with constants factored out. Sometimes this space is denoted as $H_\diamond^{1/2}$; for simplicity of notation, we simply write

$$H^{1/2}(\partial\Omega) = \left\{ f \in \mathcal{H}^{1/2}(\partial\Omega) : \int_{\partial\Omega} f \,\mathrm{d}\sigma = 0 \right\}.$$

The corresponding dual space is denoted by $H^{-1/2}(\partial\Omega)$, and the associated dual pairing in $H^{-1/2} \times H^{1/2}$ is denoted by $\langle \cdot\,, \cdot \rangle$.

Given $f \in H^{1/2}$, the associated boundary value problem

$$\Delta u_{1,f} = 0 \quad \text{in } \Omega, \qquad u_{1,f} = f \quad \text{on } \partial\Omega$$

has a unique solution in $H^1(\Omega)$. Moreover, by the Poincaré inequality, the mapping $f \to \nabla u_{1,f}$ is injective from $H^{1/2}(\partial\Omega) \to L^2(\Omega)^n$, such that we may define the $H^{1/2}(\partial\Omega)$-norm in this paper as

$$\|f\|_{H^{1/2}(\partial\Omega)} := \|\nabla u_{1,f}\|_{L^2(\Omega)^n}.$$

Moreover, we define $u_{\gamma,f}$ as the solution in $H^1(\Omega)$ of the problem (1.1):

$$\operatorname{div}(\gamma\nabla u_{\gamma,f}) = 0 \quad \text{in } \Omega, \qquad u_{\gamma,f} = f \quad \text{on } \partial\Omega.$$

For a given conductivity γ satisfying (1.2), the DtN operator Λ_γ in (1.3) is a continuous operator. It is well known that the DtN operators can be rewritten in terms of energy integrals: defining F by (1.4), we have for γ_1 and γ_2 satisfying (1.2) that (cf. [13])

$$F(\gamma_1) - F(\gamma_2) = \Lambda_{\gamma_1} - \Lambda_{\gamma_2}, \tag{2.1}$$

with

$$\langle [\Lambda_{\gamma_1} - \Lambda_{\gamma_2}]f, g\rangle = \int_\Omega (\gamma_1 - \gamma_2)\nabla u_{\gamma_1,f} \cdot \nabla u_{\gamma_2,g}\, \mathrm{d}x, \quad \forall\, f, g \in H^{1/2}(\partial\Omega). \tag{2.2}$$

Moreover, the DtN map $\Lambda(\gamma)$ (and hence the parameter-to-solution map $F(\gamma)$) is Fréchet-differentiable with respect to the L^∞-norm of γ (cf., e.g., [17]), and the derivative can be expressed as

$$F'[\gamma_1]w = \Lambda'_{\gamma_1}(w), \tag{2.3}$$

$$\langle \Lambda'_{\gamma_1}(w)f, g\rangle = \int_\Omega w\nabla u_{\gamma_1,f} \cdot \nabla u_{\gamma_1,g}\, \mathrm{d}x, \quad w \in L^\infty(\Omega). \tag{2.4}$$

Our analysis is based on the following assumptions, which we assume to hold for the rest of the article.

ASSUMPTION 1.

- *We assume that we are given a (finite or infinite) sequence of orthonormal Dirichlet data $(f_i)_{i\in I}$, $f_i \in H^{1/2}(\partial\Omega)$.*
- *We set as the domain of definition of the parameter-to-data map*

$$D(F) := \{\gamma \in L^\infty(\Omega) \mid \underline{\alpha} \le \gamma \le \overline{\alpha},\ \textit{and } (2.5) \textit{ holds}\},$$

where

$$\sum_{i,j\in I} |\langle [\Lambda_\gamma - \Lambda_1]f_i, f_j\rangle|^2 < \infty. \tag{2.5}$$

The first assumption is not much of a restriction, as the Dirichlet data are part of the experimental design and can be chosen to be orthonormalized. Also the second one is not severe because (2.5) holds if we have only finitely many Dirichlet data or, in the case of infinitely many f_i, if the deviations from the background conductivity $\delta\gamma$ have a common compact support inside of Ω. (By a standard inclusion estimate, e.g., [3, 12], this implies that the eigenvalues of $\Lambda_\gamma - \Lambda_1$ are exponentially decaying and hence summable). This is also a usual assumption in the impedance tomography problem. We note that we do not a priori require that the f_i form a complete basis in $H^{1/2}(\partial\Omega)$. Thus, much of our analysis is also valid in the case of finitely many measurements or measurements on only a part of the boundary.

Associated to the set of Dirichlet data, we set

$$V_D := \text{span}(f_i)_{i\in I} \subset H^{1/2}(\partial\Omega).$$

The quadratic form $\langle(\Lambda_\gamma - \Lambda_1)f, g\rangle$ associated to the DtN mapping defines a linear operator $\Lambda_\gamma - \Lambda_1 : V_D \to V_D'$. Hence, the parameter-to-solution map can be defined as a mapping

$$F : D(F) \subset X := L^\infty(\Omega) \to Y := L(V_D, V_D'),$$
$$\gamma \to \Lambda_\gamma - \Lambda_1,$$

with the norm

$$\|F(\gamma)\|_Y^2 : = \sum_{i,j\in I} |\langle[\Lambda_\gamma - \Lambda_1]f_i, f_j\rangle|^2,$$

and where $L(X, Y)$ denotes the space of bounded linear operators from $X \to Y$. Introducing the Riesz isomorphism $\mathcal{I} : H^{-1/2} \to H^{1/2}$ and P_{V_D} the orthogonal projection onto V_D, we may write the norm

$$\|F(\gamma)\|_Y^2 = \sum_{i,j\in I} \left|(\mathcal{I}[\Lambda_\gamma - \Lambda_1]f_i, f_j)_{H^{1/2},H^{1/2}}\right|^2 = \|P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_1]\|^2_{\text{HS}(V_D)}$$

as the Hilbert–Schmidt norm of $P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_1]$ for the operator mapping from $V_D \to V_D$. Thus, the image space is equipped with a Hilbert space structure. Note that for self-adjoint compact operators, the Hilbert–Schmidt norm is the sum of squares of the eigenvalues. We will denote by $\|\cdot\|_{\text{HS}}$ the Hilbert–Schmidt norm (omitting the underlying spaces) and by $\|\cdot\|_2$ the operator norm for a linear operator from $X \to Y$. Moreover, for functions in L^∞, we set $\|\cdot\|_\infty = \|\cdot\|_{L^\infty}$.

REMARK 2.1. By our choice of the $H^{1/2}$-norm, the Riesz isomorphism $\mathcal{I}$ is nothing but the inverse of the DtN operator Λ_1 (i.e., the Neumann-to-Dirichlet map): Indeed, by definition, $z = \mathcal{I}g$ holds if and only if, for all $f \in H^{1/2}$,

$$(z, f)_{H^{1/2},H^{1/2}} = \langle g, f\rangle_{H^{-1/2},H^{1/2}}.$$

The left-hand side can be written by the variational formulation of $\Delta u_{1,z} = 0$, $(\partial/\partial n)u_{1,z} = \Lambda_1 z$ as

$$(z, f)_{H^{1/2},H^{1/2}} = (\nabla u_{1,z}, \nabla u_{1,f})_{L^2(\Omega)^n} = \langle\Lambda_1 z, f\rangle_{H^{-1/2},H^{1/2}},$$

which yields that $z = \Lambda_1^{-1}g$.

For later use we also recall the Löwner ordering for self-adjoint operators on a Hilbert space H: we have

$$A \leq_L B \quad \Longleftrightarrow \quad (Ax, x)_H \leq (Bx, x)_H, \quad \forall\, x \in H.$$

A useful property of this ordering is that, for Hilbert–Schmidt operators,

$$0 \leq A \leq_L B \quad \Longrightarrow \quad \|A\|_{\text{HS}} \leq \|B\|_{\text{HS}}. \tag{2.6}$$

This follows from Weyl's inequality for the eigenvalues and since the Löwner ordering implies a corresponding eigenvalue inequality. Moreover, the following result with an arbitrary bounded operator T and T^* its adjoint will be used frequently:

$$A \leq_L B \quad \Longrightarrow \quad T^*AT \leq_L T^*BT. \tag{2.7}$$

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3. Löwner convexity results for F. Before we state the main convexity results, we give the following well-known monotonicity result of the Fréchet derivative; see, e.g., [10, equation (2.3)]:

LEMMA 3.1. *Let $\gamma \in D(F)$. If $w(x) \geq 0$ a.e., $w \in L^\infty(\Omega)$, then $\Lambda_\gamma'(w) \geq_L 0$. In particular, for $w_1, w_2 \in L^\infty(\Omega)$ with $0 \leq w_1 \leq w_2$ a.e. in Ω, and any $\gamma \in D(F)$, we have the estimate*

$$\|\Lambda_\gamma'(w_1)\|_Y \leq \|\Lambda_\gamma'(w_2)\|_Y. \tag{3.1}$$

A main tool for estimates of the tangential cone condition is the following classical structural convexity property of the forward map. According to Harrach and Ullrich [10], the result goes back to Ikehata [11] and Kang, Seo, and Sheen [15]. In full generality, the statement can be found in the work of Harrach and Ullrich [10, Lemma 3.1] and Harrach and Seo [9, Lemma 2.1], where it is written in terms of the Neumann-to-Dirichlet operator, which, however, can be easily translated to our setup.

THEOREM 3.2. *For any $\gamma, \gamma^\dagger \in D(F)$, we have*

$$0 \leq_L \Lambda(\gamma) - \Lambda(\gamma^\dagger) - \Lambda_\gamma'(\gamma - \gamma^\dagger) \leq_L \Lambda_\gamma'\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma^\dagger}\right). \tag{3.2}$$

We provide a new proof of this known result based on operator-theoretic ideas, which also allows us to state a slight improvement of the upper bound. Essential in our proof is Lemma 3.3 below, which might be of interest in itself since it relates gradients of solutions of (1.1) using simple projection and multiplication operators.

We need a little bit of notation. Define the operator S_γ as the solution operator for the differential equation in (1.1) with homogeneous Dirichlet condition and given right-hand side, i.e., for γ satisfying (1.2), we set

$$S_\gamma : H^{-1}(\Omega) \to H_0^1(\Omega),$$
$$h \to v,$$

where v is the solution of

$$\operatorname{div}(\gamma \nabla v) = h \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

By definition, we have that S_γ is the inverse operator to the continuous differential operator (the symbol $\bullet$ indicates a placeholder)

$$\operatorname{div}(\gamma \nabla \bullet) : H_0^1(\Omega) \to H^{-1}(\Omega), \tag{3.3}$$

i.e., we have

$$\operatorname{div}(\gamma \nabla \bullet) S_\gamma = I_{H^{-1} \to H^{-1}}, \quad S_\gamma \operatorname{div}(\gamma \nabla \bullet) = I_{H_0^1 \to H_0^1}, \tag{3.4}$$

where $I_{X \to X}$ denotes the identity operator on X.

Furthermore, for a function $\kappa \in L^\infty(\Omega)$, we define the multiplication operator

$$M_\kappa : L^2(\Omega)^n \to L^2(\Omega)^n, \quad \vec{f}(x) \to \kappa(x)\vec{f}(x). \tag{3.5}$$

Note that the multiplication operator satisfies $M_{\kappa_1} M_{\kappa_2} = M_{\kappa_1 \kappa_2}$, these operators commute, and $\|M_\kappa\|_2 \leq \|\kappa\|_\infty$.

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With this notation, we state the following useful lemma.

LEMMA 3.3. *For any γ_1 and γ_2 satisfying (1.2) and any $f \in H^{1/2}(\partial\Omega)$, we have that the operator*

$$K : L^2(\Omega)^n \to L^2(\Omega)^n, \quad K := I - \nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet] \tag{3.6}$$

has a continuous inverse given by

$$K^{-1} = I + \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]. \tag{3.7}$$

Moreover, we have that the continuous operator $Q_\gamma : L^2(\Omega)^n \to L^2(\Omega)^n$ defined by

$$Q_\gamma := M_{\sqrt{\gamma}} \nabla S_\gamma \operatorname{div}[M_{\sqrt{\gamma}} \bullet] \tag{3.8}$$

is an orthogonal projection operator, and the range of Q_γ is orthogonal to the space

$$\{z \in L^2(\Omega)^n \mid \operatorname{div}(\gamma^{1/2} z) = 0\}. \tag{3.9}$$

Finally, we have that

$$\begin{aligned} \nabla u_{\gamma_2,f} &= \left(I - \left(M^{-1}_{\sqrt{\gamma_2}} Q_{\gamma_2} M^{-1}_{\sqrt{\gamma_2}}\right) M_{\gamma_2 - \gamma_1}\right)^{-1} \nabla u_{\gamma_1,f} \\ &= \left(I + \left(M^{-1}_{\sqrt{\gamma_1}} Q_{\gamma_1} M^{-1}_{\sqrt{\gamma_1}}\right) M_{\gamma_2 - \gamma_1}\right) \nabla u_{\gamma_1,f}. \end{aligned} \tag{3.10}$$

Proof. Let K and K^{-1} be the continuous operators in (3.6) and (3.7), respectively. We verify that $KK^{-1} = \mathrm{Id}_{L^2(\Omega)^n \to L^2(\Omega)^n}$. Indeed,

$$KK^{-1} = I + \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet] - \nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet] \tag{3.11}$$

$$- (\nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1) \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]]). \tag{3.12}$$

Using (3.4), the last line (3.12) can be simplified to

$$\begin{aligned} &- (\nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1) \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]]) \\ &\quad = - (\nabla S_{\gamma_2} \operatorname{div}[\gamma_2 \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]] - \nabla S_{\gamma_2} \operatorname{div}[\gamma_1 \nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]]) \\ &\quad = -(\nabla S_{\gamma_1} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet] - \nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1) \bullet]). \end{aligned}$$

Thus, this cancels with the terms in (3.11), yielding $KK^{-1} = I$. In a similar manner we obtain $K^{-1}K = I$, which proves that these operators are continuously invertible and inverse to each other.

Next, we verify that Q_γ is idempotent and self-adjoint:

$$Q_\gamma Q_\gamma = M_{\sqrt{\gamma}} \nabla S_\gamma \operatorname{div}[\gamma \nabla S_\gamma \operatorname{div}[M_{\sqrt{\gamma}} \bullet]] = M_{\sqrt{\gamma}} \nabla S_\gamma \operatorname{div}[M_{\sqrt{\gamma}} \bullet],$$

where we again used (3.4). The operator is easily seen to be self-adjoint since $(\operatorname{div})^* = -\nabla$, and both S_γ and M are self-adjoint. Thus, Q_γ is an orthogonal projector. Let z be in the set (3.9). By the weak definition of div, we have that $(\sqrt{\gamma} z, \nabla \phi)_{L^2(\Omega)^n} = 0$ for all $\phi \in H_0^1$. Thus, for arbitrary $v \in L^2(\Omega)^n$,

$$(Q_\gamma v, z)_{L^2(\Omega)^n} = -(\nabla S_\gamma \operatorname{div}[M_{\sqrt{\gamma}} v], \sqrt{\gamma} z)_{L^2(\Omega)^n} = 0,$$

since S_γ maps into H_0^1. This proves the statement about the range of $Q_\gamma v$.

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Finally, we verify (3.10). By definition of the inhomogeneous Dirichlet problem, we have the identity $u_{\gamma_2,f} = u_{\gamma_1,f} + w$, where w satisfies the homogeneous problem

$$\operatorname{div}(\gamma_2 \nabla w) = -\operatorname{div}(\gamma_2 \nabla u_{\gamma_1,f}) = -\operatorname{div}((\gamma_2 - \gamma_1)\nabla u_{\gamma_1,f}),$$

and where we used that $u_{\gamma_2,f}$ solves the problem (1.1) with $\gamma = \gamma_2$. Thus,

$$u_{\gamma_2,f} = [I - S_{\gamma_2} \operatorname{div}((\gamma_2 - \gamma_1)\nabla)] u_{\gamma_1,f}.$$

Applying the gradient yields

$$\nabla u_{\gamma_2,f} = (I - \nabla S_{\gamma_2} \operatorname{div}[(\gamma_2 - \gamma_1)\bullet])\nabla u_{\gamma_1,f} = (I - \nabla S_{\gamma_2} \operatorname{div}[M_{\gamma_2-\gamma_1}\bullet])\nabla u_{\gamma_1,f}.$$

By the previous results, the operator on the right-hand side is invertible, which, together with (3.8), yields the result. □

Proof of Theorem 3.2. Define

$$B(\gamma, \gamma^\dagger) := \Lambda(\gamma) - \Lambda(\gamma^\dagger) - \Lambda'_\gamma(\gamma - \gamma^\dagger). \tag{3.13}$$

Then, from (2.1) and (2.3), it follows that

$$\langle B(\gamma, \gamma^\dagger) f, f\rangle = \int_\Omega (\gamma - \gamma^\dagger)\nabla u_{\gamma,f} \cdot (\nabla u_{\gamma^\dagger,f} - \nabla u_{\gamma,f})\,\mathrm{d}x.$$

Using (3.10) with $\gamma_2 = \gamma^\dagger$ and $\gamma_1 = \gamma$ and (3.5), we write this as

$$\begin{aligned}
\langle B(\gamma, \gamma^\dagger) f, f\rangle &= \big(M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}, \big[\big(I - M^{-1}_{\sqrt{\gamma^\dagger}} Q_{\gamma^\dagger} M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma^\dagger-\gamma}\big) - I\big]\nabla u_{\gamma,f}\big)_{L^2(\Omega)^n} \\
&= \big(M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}, M^{-1}_{\sqrt{\gamma^\dagger}} Q_{\gamma^\dagger} M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}\big)_{L^2(\Omega)^n} \\
&= \big(M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}, Q_{\gamma^\dagger} M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}\big)_{L^2(\Omega)^n}.
\end{aligned} \tag{3.14}$$

Since $Q_{\gamma^\dagger}$ is an orthogonal projector, it satisfies $0 \le_L Q_{\gamma^\dagger} \le_L I$. Inserting these inequalities and making use of (2.7) with $T = M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}$ yields the upper and lower bounds. Note that the upper bound reads

$$\begin{aligned}
&\big(M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}, M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}\big)_{L^2(\Omega)^n} \\
&\quad = \int_\Omega \frac{(\gamma - \gamma^\dagger)^2}{\gamma^\dagger} |\nabla u_{\gamma,f}|^2\,\mathrm{d}x = \Lambda'_\gamma\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma^\dagger}\right).
\end{aligned}$$ □

By rearranging terms and switching γ and $\gamma^\dagger$ in (3.2) using the lower and upper bounds, we obtain the following two inequalities (cf. [9, 10]):

$$\Lambda'_\gamma(\gamma - \gamma^\dagger) \le_L \Lambda(\gamma) - \Lambda(\gamma^\dagger) \le_L \Lambda'_{\gamma^\dagger}(\gamma - \gamma^\dagger), \tag{3.15}$$

$$\Lambda'_{\gamma^\dagger}\left(\frac{\gamma^\dagger}{\gamma}(\gamma - \gamma^\dagger)\right) \le_L \Lambda(\gamma) - \Lambda(\gamma^\dagger) \le_L \Lambda'_\gamma\left(\frac{\gamma}{\gamma^\dagger}(\gamma - \gamma^\dagger)\right). \tag{3.16}$$

REMARK 3.4. We may observe by a Taylor expansion that the middle term in (3.2) can be written as

$$-\tfrac{1}{2}\Lambda''_\gamma(\gamma^\dagger - \gamma, \gamma^\dagger - \gamma) + o(\|\gamma^\dagger - \gamma\|^2_\infty),$$

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while the right-hand side is

$$\Lambda'_{\gamma^\dagger}(|\gamma^\dagger - \gamma|^2/\gamma^\dagger) + o(\|\gamma^\dagger - \gamma\|_\infty^2).$$

By setting $\gamma = \gamma^\dagger + \epsilon w$ for arbitrary $w \in L^\infty$, and taking the limit, we obtain

$$0 \leq -\Lambda''_{\gamma^\dagger}(w, w) \leq 2\Lambda'_{\gamma^\dagger}\left(\frac{|w|^2}{\gamma^\dagger}\right). \tag{3.17}$$

Hence the second derivative of F is always negative definite. A similar negativity result can be obtained for all even-order derivatives.

For later use, we also establish the following related inequality.

LEMMA 3.5. *Let $\gamma, \gamma^\dagger \in D(F)$ and $\xi_\dagger := \|(\gamma - \gamma^\dagger)/\gamma^\dagger\|_\infty \leq 2$. Then we have the estimates*

$$0 \leq_L \Lambda'_{\gamma^\dagger}(\gamma - \gamma^\dagger) - \Lambda'_\gamma(\gamma - \gamma^\dagger) \leq_L (2 + \xi_\dagger)\Lambda'_\gamma\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma^\dagger}\right). \tag{3.18}$$

Proof. Define $\mathcal{A} := \Lambda'_{\gamma^\dagger}(\gamma-\gamma^\dagger) - \Lambda'_\gamma(\gamma-\gamma^\dagger)$. Using Lemma 3.3 with $\gamma_1 = \gamma$, $\gamma_2 = \gamma^\dagger$, and the shortcut notation

$$M_{\Delta\gamma} := M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger} = M_{(\gamma-\gamma^\dagger)/\sqrt{\gamma^\dagger}}, \tag{3.19}$$

we find

$$\begin{aligned}
\langle \mathcal{A}f, f\rangle &= \int (\gamma - \gamma^\dagger)(|\nabla u_{\gamma^\dagger,f}|^2 - |\nabla u_{\gamma,f}|^2)\,\mathrm{d}x \\
&= \big(M_{\gamma-\gamma^\dagger}\big(I + M^{-1}_{\sqrt{\gamma^\dagger}} Q_{\gamma^\dagger} M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\big)\nabla u_{\gamma,f}, \big(I + M^{-1}_{\sqrt{\gamma^\dagger}} Q_{\gamma^\dagger} M^{-1}_{\sqrt{\gamma^\dagger}} M_{\gamma-\gamma^\dagger}\big)\nabla u_{\gamma,f}\big) \\
&\quad - \big(M_{\gamma-\gamma^\dagger}\nabla u_{\gamma,f}, \nabla u_{\gamma,f}\big) \\
&= \big(\big[2M_{\Delta\gamma}Q_{\gamma^\dagger}M_{\Delta\gamma} + M_{\Delta\gamma}Q_{\gamma^\dagger}M_{\Delta\gamma/\sqrt{\gamma^\dagger}}Q_{\gamma^\dagger}M_{\Delta\gamma}\big]\nabla u_{\gamma,f}, \nabla u_{\gamma,f}\big) \\
&= \big(\big[2I + M_{\Delta\gamma/\sqrt{\gamma^\dagger}}\big]Q_{\gamma^\dagger}M_{\Delta\gamma}\nabla u_{\gamma,f}, Q_{\gamma^\dagger}M_{\Delta\gamma}\nabla u_{\gamma,f}\big),
\end{aligned}$$

where we used that $Q^2_{\gamma^\dagger} = Q_{\gamma^\dagger}$ and all operators are self-adjoint. Since $\xi_\dagger = \|\Delta\gamma/\sqrt{\gamma^\dagger}t\|$, we find that

$$0 \leq_L (2 - \xi_\dagger)I \leq_L 2I + M_{\Delta\gamma/(\gamma^\dagger)^{1/2}} \leq_L (2 + \xi_\dagger)I,$$

and the lower bound follows as well as the upper bound since $Q^2_{\gamma^\dagger} = Q_{\gamma^\dagger} \leq_L I$. ∎

Improved upper bound. Although it is not needed for the main results, we note that the upper bound in Theorem 3.2 can be strengthened by a more detailed analysis. The following is an improvement of the upper bound in (3.2).

THEOREM 3.6. *For $\gamma, \gamma^\dagger \in D(F)$ we have*

$$\langle[\Lambda_\gamma - \Lambda_{\gamma^\dagger} - \Lambda'_\gamma(\gamma - \gamma^\dagger)]f, f\rangle \leq \inf_{w \in L^2(\Omega)^n : \mathrm{div}(\sqrt{\gamma^\dagger} w) = 0} \|M_{(\gamma-\gamma^\dagger)/\sqrt{\gamma^\dagger}}\nabla u_{\gamma,f} - w\|^2.$$

Moreover, the following estimate holds:

$$\begin{aligned}
&\langle[\Lambda_\gamma - \Lambda_{\gamma^\dagger} - \Lambda'_\gamma(\gamma - \gamma^\dagger)]f, f\rangle \\
&\leq \left\langle \Lambda'_\gamma\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma^\dagger}\right)f, f\right\rangle - \langle[P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}]^\dagger P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]f, P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]f\rangle.
\end{aligned} \tag{3.20}$$

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Note that $[P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}]^\dagger$ is the pseudoinverse of $[P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}]$. In the case of full measurements, this agrees with $(\mathcal{I}\Lambda_{\gamma^\dagger})^{-1}$.

Proof. We use the notation in (3.13) and (3.19). From (3.14) and as $Q_{\gamma^\dagger}$ is an orthogonal projector, we find that

$$\langle B(\gamma,\gamma^\dagger)f,f\rangle = \|Q_{\gamma^\dagger}M_{\Delta\gamma}\nabla u_{\gamma,f}\|^2 = \inf_{w\in \operatorname{Im}(Q_{\gamma^\dagger})^\perp} \|M_{\Delta\gamma}\nabla u_{\gamma,f} - w\|^2, \tag{3.21}$$

where $\operatorname{Im}(Q_{\gamma^\dagger})^\perp$ is the orthogonal complement of the range of $Q_{\gamma^\dagger}$. According to (3.9), the set of w with $\operatorname{div}((\gamma^\dagger)^{1/2}w)$ is a subset of $Q_{\gamma^\dagger}$, which yields the first inequality.

In order to verify (3.20), we take $w = w_c$ as

$$w_c = \sum_{i=1}^{\infty} c_i (\gamma^\dagger)^{1/2}\nabla u_{\gamma^\dagger,f_i},$$

where $c_i \in \ell^2$ are coefficients to be specified below. Note that $\operatorname{div}((\gamma^\dagger)^{1/2}w_c) = 0$ since $\nabla u_{\gamma^\dagger,f_i}$ solves (1.1) with $\gamma = \gamma^\dagger$. With the definition $f_c = \sum_{i\in I} c_i f_i$ and after expanding the square, we find that

$$\begin{aligned}\|M_{\Delta\gamma}\nabla u_{\gamma,f} - w_c\|^2 &= \int M_{\Delta\gamma}^2|\nabla u_{\gamma,f}|^2\,\mathrm{d}x - 2\sum_{i=1}^{\infty} c_i \int_\Omega (\gamma-\gamma^\dagger)\nabla u_{\gamma,f}\cdot\nabla u_{\gamma^\dagger,f_i}\,\mathrm{d}x \\ &\quad + \sum_{i,j=1}^{\infty} c_i c_j \int \gamma^\dagger \nabla u_{\gamma^\dagger,f_i}\cdot\nabla u_{\gamma^\dagger,f_i}\,\mathrm{d}x \\ &= \langle \Lambda_\gamma'((\Delta\gamma)^2)f,f\rangle - 2\langle \Lambda_\gamma - \Lambda_{\gamma^\dagger}f, f_c\rangle + \langle \Lambda_{\gamma^\dagger}f_c, f_c\rangle. \end{aligned} \tag{3.22}$$

The last line can be rewritten as

$$\langle \Lambda_\gamma'((\Delta\gamma)^2)f,f\rangle - 2(\mathcal{I}(\Lambda_\gamma - \Lambda_{\gamma^\dagger})f, P_{V_D}f_c)_{H^{1/2}} + (\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}f_c, P_{V_D}f_c)_{H^{1/2}},$$

because $f_c = P_{V_D}f_c$ by definition. Thus, minimizing over f_c yields an upper bound for (3.21), and the minimizer satisfies the optimality condition

$$P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}f_c = P_{V_D}\mathcal{I}(\Lambda_\gamma - \Lambda_{\gamma^\dagger})f.$$

Inserting this into (3.22) yields the result. ∎

A consequence of Theorem 3.6 is the following result for the error norm.

THEOREM 3.7. *Let the same assumptions as in Theorem 3.6 hold. Then there exists a constant C depending only on $\underline{\alpha}$, $\overline{\alpha}$, and Ω such that*

$$\|\Lambda_\gamma - \Lambda_{\gamma^\dagger}\|_Y^2 \le C\sum_{i\in I}\left\langle \Lambda_\gamma'\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)f_i, f_i\right\rangle. \tag{3.23}$$

Proof. Standard elliptic estimates yield with some constants that depend on $\underline{\alpha}$, $\overline{\alpha}$, and Ω that

$$c_1\|f\|_{H^{1/2}}^2 \le \langle \Lambda_{\gamma^\dagger}f,f\rangle \le c_2\|f\|_{H^{1/2}}^2.$$

Thus, for $f \in V_D$ and since P_{V_D} is an orthogonal projector, we have that

$$\|f\|_{H^{1/2}}^2 \le c_2\langle \Lambda_{\gamma^\dagger}P_{V_D}f, P_{V_D}f\rangle = c_2(\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}f, P_{V_D}f) = c_2(P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}f,f).$$

As a consequence, the operator $P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}$ is continuously invertible on V_D and

$$([P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}]^\dagger P_{V_D}f, P_{V_D}f) \leq c_2\|P_{V_D}f\|^2_{H^{1/2}}. \tag{3.24}$$

As a consequence of the fact that the right-hand side in (3.20) is positive, we obtain that

$$\langle [P_{V_D}\mathcal{I}\Lambda_{\gamma^\dagger}P_{V_D}]^\dagger P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]f, P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]f\rangle \leq \left\langle \Lambda'_\gamma\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)f, f\right\rangle.$$

Thus, combining this with (3.24) yields, for $f \in V_D$,

$$\|P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]f\|^2_{H^{1/2}} \leq \left\langle \Lambda'_\gamma\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)f, f\right\rangle.$$

It is not difficult to see that $P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]$ is a self-adjoint operator in V_D, which is also a Hilbert–Schmidt operator by our assumptions. Thus this operator has a countable set of real eigenvalues λ_i with associated eigenvectors h_i that form an orthonormal system. The Frobenius norm can be rewritten as

$$\begin{aligned}\|P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]\|^2_{\mathrm{HS}(V_D)} = \sum_i \lambda_i^2 &= \sum_i \|P_{V_D}\mathcal{I}[\Lambda_\gamma - \Lambda_{\gamma^\dagger}]h_i\|^2_{H^{1/2}} \\ &\leq \sum_i \left\langle \Lambda'_\gamma\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)h_i, h_i\right\rangle.\end{aligned}$$

The right-hand side can be interpreted as the trace norm of $P_{V_D}\mathcal{I}\Lambda'_\gamma(|\gamma-\gamma^\dagger|^2/\gamma^\dagger)$, and thus the h_i can be replaced by any orthonormal basis, in particular by f_i, which completes the proof. □

REMARK 3.8. We note that the norm estimates (3.1) and (3.23) remain valid if the Y-norm is replaced by the operator norm $\|\cdot\|_{L(H^{1/2},H^{-1/2})}$. This follows from the fact that the inequalities are derived from the Löwner ordering and hold in particular for the largest eigenvalues, which agree with the operator norm.

4. Tangential cone conditions. We can now state some sufficient conditions for the tangential cone conditions. As stated in the introduction, we aim to verify the tangential cone conditions not for all elements in a neighborhood but for specific elements γ and $\gamma^\dagger$. If the iterates of a Landweber iteration satisfy these conditions, then convergence can be proven.

We organize our results in three classes: conditions based on (i) source conditions, (ii) finite-dimensionality, and (iii) monotonicity.

In terms of the forward operator, the upper bound in (3.2) (making additional use of the nonnegativity in (3.2)) can be written as

$$\|F(\gamma) - F(\gamma^\dagger) - F'[\gamma](\gamma-\gamma^\dagger)\|_Y \leq \left\|F'[\gamma]\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)\right\|_Y, \tag{4.1}$$

which obviously shows the helpfulness of finding conditions for the tangential cone conditions. Using this inequality yields the following result, which serves as the basis for all the following sufficient conditions.

THEOREM 4.1. *Let $\gamma, \gamma^\dagger \in D(F)$.*

1. If, for some $\zeta < 1$, it holds that

$$\left\|F'[\gamma]\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)\right\|_Y \leq \zeta\|F'[\gamma](\gamma-\gamma^\dagger)\|_Y, \tag{4.2}$$

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then the strong tangential cone condition with $\eta = \zeta/(1-\zeta)$ is satisfied. If the same condition holds with $\zeta = 1$, then the weak tangential cone condition with $\eta \geq \frac{1}{2}$ is satisfied.

2. *Let $\xi_\dagger = \|(\gamma - \gamma^\dagger)/\gamma^\dagger\|_\infty \leq 2$ be satisfied. If for some*

$$\theta_\eta < \theta_* := \underline{\alpha}\,\frac{3+\xi_\dagger}{4+\xi_\dagger}$$

it holds that

$$\|F'[\gamma^\dagger](|\gamma-\gamma^\dagger|^2)\|_Y \leq \theta_\eta \|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y, \tag{4.3}$$

then the strong tangential cone condition is satisfied with this

$$\eta = (3+\xi_\dagger)\frac{\theta_\eta/\underline{\alpha}}{1-\theta_\eta/\underline{\alpha}}.$$

Proof. Assume that (4.2) holds with $\zeta < 1$. Thus,

$$\begin{aligned}
\|F'[\gamma](\gamma-\gamma^\dagger)\|_Y &\leq \|F'[\gamma](\gamma-\gamma^\dagger) - (F(\gamma)-F(\gamma^\dagger))\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y \\
&\leq \left\|F'[\gamma]\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)\right\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y \\
&\leq \zeta\|F'[\gamma](\gamma-\gamma^\dagger)\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y.
\end{aligned}$$

From this, we conclude that

$$\|F'[\gamma](\gamma-\gamma^\dagger)\|_Y \leq \frac{1}{1-\zeta}\|F(\gamma)-F(\gamma^\dagger)\|_Y.$$

Thus, with (4.1),

$$\begin{aligned}
\|F(\gamma)-F(\gamma^\dagger)-F'[\gamma](\gamma-\gamma^\dagger)\|_Y &\leq \left\|F'[\gamma]\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma^\dagger}\right)\right\|_Y \\
&\leq \zeta\|F'[\gamma](\gamma-\gamma^\dagger)\|_Y \leq \frac{\zeta}{1-\zeta}\|(F(\gamma)-F(\gamma^\dagger)\|_Y.
\end{aligned}$$

In the case of $\zeta = 1$, we derive the weak cone condition for $\eta \geq \frac{1}{2}$ easily from (1.5) and the upper estimate (3.2).

Consider now the case that (4.3) holds. We have

$$\begin{aligned}
\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y &\leq \|F'[\gamma^\dagger](\gamma-\gamma^\dagger) + (F(\gamma^\dagger)-F(\gamma))\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y \\
&\leq \left\|F'[\gamma^\dagger]\left(\frac{|\gamma-\gamma^\dagger|^2}{\gamma}\right)\right\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y \\
&\leq \left\|\frac{1}{\gamma}\right\|_\infty \|F'[\gamma^\dagger](|\gamma-\gamma^\dagger|^2)\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y \\
&\leq \frac{1}{\underline{\alpha}}\theta_\eta\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y + \|F(\gamma)-F(\gamma^\dagger)\|_Y.
\end{aligned}$$

Thus,

$$\left(1-\frac{\theta_\eta}{\underline{\alpha}}\right)\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y \leq \|F(\gamma)-F(\gamma^\dagger)\|_Y.$$

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Using (3.18), with γ and $\gamma^\dagger$ swapped and the constant $C = 2 + \xi_\dagger$, we obtain the strong tangential cone condition as follows:

$$\begin{aligned}
&\|F(\gamma) - F(\gamma^\dagger) - F'[\gamma](\gamma^\dagger - \gamma)\|_Y \\
&\quad \le \|F(\gamma) - F(\gamma^\dagger) - F'[\gamma^\dagger](\gamma^\dagger - \gamma)\|_Y + \|F'[\gamma^\dagger](\gamma^\dagger - \gamma) - F'[\gamma](\gamma^\dagger - \gamma)\|_Y \\
&\quad \le \left\| F'[\gamma^\dagger]\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma}\right)\right\|_Y + C\left\| F'[\gamma^\dagger]\left(\frac{|\gamma - \gamma^\dagger|^2}{\gamma}\right)\right\|_Y \\
&\quad \le (1 + C)\left\|\frac{1}{\gamma}\right\|_\infty \|F'[\gamma^\dagger]|\gamma - \gamma^\dagger|^2\|_Y \\
&\quad \le (3 + \xi_\dagger)\frac{1}{\underline{\alpha}}\theta_\eta \|F'[\gamma^\dagger]\gamma - \gamma^\dagger\|_Y \\
&\quad \le (3 + \xi_\dagger)\frac{\theta_\eta/\underline{\alpha}}{1 - \theta_\eta/\underline{\alpha}}\|F(\gamma) - F(\gamma^\dagger)\|_Y. \qquad \square
\end{aligned}$$

Tangential cone conditions by source conditions. An immediate corollary of the previous theorem is the result that the cone conditions are satisfied if a source condition (or a conditional stability estimate) holds.

COROLLARY 4.2. *Let W be a Hilbert space that is continuously embedded into L^∞. Assume that a source condition*

$$\gamma - \gamma^\dagger = (F'[\gamma^\dagger]^* F'[\gamma^\dagger])^\mu \omega$$

holds with $\mu > \frac{1}{2}$, where $F'[\gamma^\dagger]^*$ is the adjoint in W. Then, for $\|\gamma - \gamma^\dagger\|_\infty$ sufficiently small, the strong tangential cone condition holds for a given $\eta \le \frac{1}{2}$.

Proof. The left-hand side of (4.3) is bounded by

$$\|F'[\gamma^\dagger](|\gamma - \gamma^\dagger|^2)\|_Y \le L\|\gamma - \gamma^\dagger\|_\infty^2 \le L\|\gamma - \gamma^\dagger\|_W^2, \quad L = \|F'[\gamma^\dagger]\|_{2,L(L^\infty,Y)}.$$

The source condition implies a stability estimate (see, e.g., [6, p. 59])

$$\|\gamma - \gamma^\dagger\|_W \le \|\omega\|^{1/(1+2\mu)}\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)\|_Y^{2\mu/(2\mu+1)}.$$

Combining the inequalities yields

$$\begin{aligned}
\|F'[\gamma^\dagger](|\gamma - \gamma^\dagger|^2)\|_Y &\le L\|\omega\|^{2/(1+2\mu)}\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)\|_Y^{4\mu/(2\mu+1)-1}\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)\|_Y \\
&\le L\|\omega\|^{2/(1+2\mu)}(L\|\gamma - \gamma^\dagger\|_\infty)^{(2\mu-1)/(2\mu+1)}\|F'[\gamma^\dagger](\gamma - \gamma^\dagger)\|_Y.
\end{aligned}$$

If $\mu > \frac{1}{2}$ then (4.3) is verified for

$$\|\gamma - \gamma^\dagger\|_\infty \le \left(\frac{\theta_*}{L\|\omega\|^{2/(1+2\mu)}}\right)^{(2\mu+1)/(2\mu-1)}, \quad \|\gamma - \gamma^\dagger\|_\infty < \underline{\alpha}. \qquad \square$$

Tangential cone conditions by finite-dimensionality. Because of (3.1), we may estimate the left-hand side in (4.3) by

$$\|F'[\gamma^\dagger](|\gamma - \gamma^\dagger|^2)\|_Y \le \|\gamma - \gamma^\dagger\|_\infty \|F'[\gamma^\dagger](|\gamma - \gamma^\dagger|)\|_Y, \tag{4.4}$$

such that a sufficient condition for (4.3) is that, for some constant C, we have that

$$\|F'[\gamma^\dagger](|\gamma-\gamma^\dagger|)\|_Y \le C\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y, \tag{4.5}$$

and that additionally $\|\gamma-\gamma^\dagger\|_\infty$ is sufficiently small.

An interesting observation is that the right-hand side in this inequality generates a (semi-)norm:

PROPOSITION 4.3. *Define*

$$\|w\|_* := \|F'[\gamma^\dagger](|w|)\|_Y.$$

Then $\|w\|_*$ defines a semi-norm in L^∞.

Proof. Indeed, take $z_1, z_2 \in L^\infty$ arbitrary. Then, $0 \le |z_1(x)+z_2(x)| \le |z_1(x)|+|z_2(x)|$. Thus, the triangle inequality for $\|w\|_*$ follows from (3.1) with $w_1 = |z_1+z_2|$ and $w_2 = |z_1|+|z_2|$ and the triangle inequality for $\|\cdot\|_Y$. The norm is positive definite since

$$\|F'[\gamma^\dagger](|w|)\|^2 = \sum_{f\in I}\int_\Omega |w(x)||\nabla u_{\gamma^\dagger,f}(x)|^2 \,\mathrm{d}x \ge 0. \qquad \square$$

Consequently, condition (4.5) can be rephrased as an equivalence condition between two (semi-)norms:

$$\|w\|_* \le C\|F'[\gamma^\dagger]w\|.$$

(Note that the reverse inequality is easy to obtain.) The well-known norm equivalence in finite-dimensional spaces leads to the following result.

COROLLARY 4.4. *Let $\gamma, \gamma^\dagger \in D(F)$ with $\gamma-\gamma^\dagger$ being in a finite-dimensional space X_n, and let $F'[\gamma^\dagger]$ be injective on X_n. Then there is a dimension-dependent constant C_n such that, for all*

$$\|\gamma-\gamma^\dagger\|_\infty \le C_n,$$

the strong tangential cone condition is satisfied for some given η.

REMARK 4.5. We imposed the condition of injectivity of $F'[\gamma^\dagger]$ since it was not required to have complete measurements, i.e., that (f_i) forms a complete orthogonal basis. If this is the case, then injectivity can be verified by the well-known uniqueness results for the EIT problem.

Tangential cone conditions by monotonicity. In view of the condition (4.5), it is obvious that it holds in the case that γ is below or above the true conductivity, since then $|\gamma-\gamma^\dagger| = \pm(\gamma-\gamma^\dagger)$. Let us state this as a corollary.

COROLLARY 4.6. *Let $\gamma, \gamma^\dagger \in D(F)$, with $\|(\gamma-\gamma^\dagger)/\gamma^\dagger\|_{L^\infty} \le 2$. Assume that either*

$$\gamma(x) \le \gamma^\dagger(x) \quad \textit{or} \quad \gamma(x) \ge \gamma^\dagger(x) \quad \forall\, x\in\Omega \textit{ a.e.}$$

Then for

$$\|\gamma-\gamma^\dagger\|_\infty \le \theta_*,$$

with θ_* as in Theorem 4.1, the strong tangential cone condition is satisfied with η as in (4.3) and $\theta_\eta = \theta_*$.

In the following we generalize this monotonicity result by imposing "imbalancing conditions," i.e., that the negative part of $\gamma-\gamma^\dagger$ is dominated by the positive part or vice versa.

We denote the positive and negative parts of $\gamma - \gamma^\dagger$ by

$$(\gamma-\gamma^\dagger)^+(x) := \max\{\gamma(x)-\gamma^\dagger(x),0\} \qquad \text{and}$$
$$(\gamma-\gamma^\dagger)^-(x) := -\min\{\gamma(x)-\gamma^\dagger(x),0\},$$

such that

$$\gamma-\gamma^\dagger = (\gamma-\gamma^\dagger)^+ - (\gamma-\gamma^\dagger)^-.$$

THEOREM 4.7. *Assume that there exists a constant C or a constant $\nu < 1$ such that*

$$\|F'[\gamma^\dagger](\gamma-\gamma)^+(x)\|_Y \le C\|F'[\gamma^\dagger](\gamma-\gamma)\|_Y \tag{4.6}$$

or

$$\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)^+\|_Y \le \nu\|F'[\gamma^\dagger](\gamma-\gamma)^-\|_Y \tag{4.7}$$

(or the respective inequalities with $(\ )^+$ and $(\ )^-$ swapped) hold.

In the case (4.6), if

$$\|\gamma-\gamma^\dagger\|_\infty \le \theta_* \frac{1}{(2C+1)}$$

holds, and in the case (4.7), if

$$\|\gamma-\gamma^\dagger\|_\infty \le \frac{\theta_*}{3}(1-\nu)$$

holds, with θ_* from Theorem 4.1, then (4.3) is satisfied. In particular, under one of these conditions, the tangential cone condition holds with any given η if additionally $\|\gamma-\gamma^\dagger\|$ is sufficiently small.

Proof. Set $p(x) = (\gamma-\gamma^\dagger)^+(x)$ and $n(x) = (\gamma-\gamma^\dagger)^-(x)$. Then $\gamma-\gamma^\dagger = p-n$ and $(\gamma-\gamma^\dagger)^2 = p^2+n^2$. We have

$$\begin{aligned}
&\|F'[\gamma^\dagger](|\gamma-\gamma^\dagger|)\|_Y \\
&\quad \le \|F'[\gamma^\dagger](p+n)\|_Y \le \|F'[\gamma^\dagger]p\|_Y + \|F'[\gamma^\dagger]n\|_Y \\
&\quad \le \|F'[\gamma^\dagger]p\|_Y + \|F'[\gamma^\dagger](n-p)\|_Y + \|F'[\gamma^\dagger]p\|_Y \\
&\quad \le 2C\|F'[\gamma^\dagger](p-n)\|_Y + \|F'[\gamma^\dagger](n-p)\|_Y \\
&\quad \le (2C+1)\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y.
\end{aligned}$$

Thus,

$$\|F'[\gamma^\dagger]((\gamma-\gamma^\dagger)^2)\|_Y \le \|\gamma-\gamma^\dagger\|_\infty(2C+1)\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y,$$

and (4.3) holds. In the case of (4.7), we estimate

$$\begin{aligned}
\|F'[\gamma^\dagger](\gamma-\gamma^\dagger)\|_Y = \|F'[\gamma^\dagger]p - F'[\gamma^\dagger]n\|_Y &\ge \big|\,\|F'[\gamma^\dagger]p\|_Y - \|F'[\gamma^\dagger]n\|_Y\big| \\
&\ge (1-\nu)\|F'[\gamma^\dagger]p\|_Y.
\end{aligned}$$

Hence (4.6) is verified with $C = 1/(1-\nu)$. The result then follows from the first part with $\nu \ge 0$. $\square$

Finally, as the most constructive result, we establish the local tangential cone conditions for general C^2-conductivities with "imbalanced" positive or negative part.

THEOREM 4.8. *Assume that* $\|\gamma - \gamma^\dagger\|_{C^2(\Omega)} \leq C_1$ *and* $\|(\gamma - \gamma^\dagger)\|_\infty \leq C_2 < \theta_*/3$, *with* θ_* *from Theorem 4.1. There is a nonnegative nondecreasing function* ψ *such that, if*

$$\|(\gamma - \gamma^\dagger)^-\|_\infty \leq \psi(\|(\gamma - \gamma^\dagger)\|_\infty) \tag{4.8}$$

holds (or with the roles of $+$ *and* $-$ *swapped), then the strong tangential cone condition is satisfied with some* $\eta < 1$.

Proof. Without loss of generality we assume that $\|(\gamma - \gamma^\dagger)^+\|_\infty > \|(\gamma - \gamma^\dagger) - \|_\infty$ and hence that $\|(\gamma - \gamma^\dagger)\|_\infty = \|(\gamma - \gamma^\dagger)^+\|_\infty$. The case in which the negative and positive parts have equal norm is ruled out by the assumptions of the theorem. Let x_0 be a point in Ω where the maximum m of $\delta\gamma := (\gamma - \gamma^\dagger)^+$ is attained. Then $\delta\gamma'(x_0) = 0$, and with the C^2-bound we may find an estimate

$$(\gamma - \gamma^\dagger)^+(x) \geq m - \frac{C_1}{2}\|x - x_0\|^2.$$

Thus, for $\|x - x_0\|^2 \leq m/C_1$, we have that $(\gamma - \gamma^\dagger)^+(x) \geq m/2$. By the monotonicity result in (3.1), it follows that

$$\|F'[\gamma^\dagger](\gamma - \gamma)^+(x)\|_Y \geq \frac{\|(\gamma - \gamma^\dagger)^+\|_\infty}{2}\|F'[\gamma^\dagger]\chi_{B_{m/C_1}(x_0)}\|_Y,$$

where $B_r(x_0)$ is the ball with center x_0 and radius r, and χ denotes the characteristic function. Define

$$\kappa(m) := \inf_{B_m(x_0)\subset\Omega} \|F'[\gamma^\dagger]\chi_{B_m}(x_0)\|.$$

This defines a nonnegative and nondecreasing function. Thus,

$$\|F'[\gamma^\dagger](\gamma - \gamma)^+(x)\|_Y \geq \frac{m}{2}\kappa\left(\frac{m}{C_1}\right).$$

Define $\nu := 1 - 3C_2/\theta_*$. By the assumption in the theorem, $0 < \nu < 1$. We let

$$\psi(m) := \frac{1}{2L} m\kappa\left(\frac{m}{C_1}\right)\nu.$$

If (4.8) is satisfied with this ψ, then with $m = \|(\gamma - \gamma^\dagger)^+\|_\infty$ we have

$$\begin{aligned}\|F'[\gamma^\dagger](\gamma - \gamma)^-(x)\|_Y &\leq L\|(\gamma - \gamma)^-\|_\infty \leq \psi(\|(\gamma - \gamma)^+\|_\infty)\\ &\leq \nu\frac{m}{2}\kappa\left(\frac{m}{C_1}\right) \leq \nu\|F'[\gamma^\dagger](\gamma - \gamma)^+(x)\|_Y.\end{aligned}$$

Thus, (4.7) holds, and since

$$\frac{\theta_\eta}{3}(1 - \nu) = C_2 \geq \|(\gamma - \gamma)\|_\infty,$$

the result follows from Theorem 4.7 . ∎

The relevance of this result is that we have the tangential cone condition satisfied in an L^∞-ball if the positive (respectively negative) part of the difference of conductivities dominates the negative (respectively positive) part.

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A convexity result. We finally use the methodology to state a condition when the least-squares functional for impedance tomography is convex in certain directions. The result is as follows.

THEOREM 4.9. *The least-squares functional for the EIT problem*

$$J(\gamma) := \tfrac{1}{2}\|F(\gamma) - F(\gamma^\dagger)\|_F^2$$

is convex on the convex set

$$\mathcal{C} = \{\gamma \in D(F) \mid \Lambda'_{\gamma^\dagger}(\gamma - \gamma^\dagger) \leq_L 0\}.$$

Proof. First, we note that $D(F)$ is a convex set and the set $\mathcal{C}$ can be written as the intersection of half-spaces with $D(F)$, thus being a convex set:

$$\mathcal{C} = \bigcap_{i \in I} \left\{ \gamma : \int_\Omega \gamma(x) |\nabla u_{\gamma^\dagger, f_i}(x)|^2 \,\mathrm{d}x \leq \int_\Omega \gamma^\dagger(x) |\nabla u_{\gamma^\dagger, f_i}(x)|^2 \,\mathrm{d}x \right\} \cap D(F).$$

For $\gamma_1, \gamma_2 \in \mathcal{C}$, consider $\gamma_t := t\gamma_1 + (1-t)\gamma_2$, $t \in (0,1)$, set $\Delta\gamma = \gamma_1 - \gamma_2$, and let $h(t) := J(\gamma(t))$. Differentiating twice yields

$$h''(t) = (F''[\gamma_t](\Delta\gamma, \Delta\gamma), F(\gamma_t) - F(\gamma^\dagger)) + \|F'[\gamma_t](\Delta\gamma)\|^2.$$

Here F'' is negative definite by (3.17), and by using the upper bound in (3.15), we have that

$$h''(t) \geq \|F'[\gamma_t](\Delta\gamma)\|^2 + (F''[\gamma_t](\Delta\gamma, \Delta\gamma), F'[\gamma^\dagger](\Delta\gamma)) \geq \|F'[\gamma_t](\Delta\gamma)\|^2 \geq 0,$$

since the inner product of two negative definite operators is positive. Thus h is convex, which proves the result. □

5. Discussion. We have stated various sufficient conditions for the tangential cone conditions for the EIT problem. Probably the practically most useful results are those using monotonicity, e.g., Theorems 4.7 and 4.8.

These results might explain to some extent the convergence behavior of the Landweber iteration for the EIT problem. The fact is that the cone conditions are only required for the iterates γ_k and the true conductivity $\gamma^\dagger$. In many numerical experiments, the initial value for the iteration is often chosen to be strictly below the true conductivity; for instance, if $\gamma^\dagger$ corresponds to inclusions that have higher conductivities than the background, and thus, naturally, the initial values of the iteration are taken as that background. Then by Theorem 4.7 (respectively, Corollary 4.6), the cone condition is satisfied for the initial iterate, and the iteration will at least remain bounded. This will also be true for a certain number of the following iterations. In the opinion of the author, this is the most plausible explanation for the observed convergence of Landweber iteration for this problem. Note that for the Robin transmission problem, similar monotonicity results were use in [8] to prove global convergence of a Newton method.

However, for the EIT problem, it is not guaranteed that the assumed monotonicity between γ_k and $\gamma^\dagger$ will hold for all iterations. In a lucky case, it will hold up to a stopping index, and then the iteration appears to be convergent. However, in an unlucky case, the cone condition might be violated, with the effect that the iterates can diverge even though the stopping criterion is not yet met. This effect may mistakenly be regarded as semiconvergence, i.e., divergence by data error, although this has nothing to do with noisy data.

For a fair investigation of the convergence of the Landweber method, it would be interesting to start with a γ that has values below and above $\gamma^\dagger$ in a high number of regions, e.g., $\gamma_0 = \gamma^\dagger +$ highly oscillatory. An interesting question is whether the iterates of the Landweber method would still remain bounded in this case.

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