

RELATIVE PERTURBATION $\tan \Theta$ -THEOREMS FOR DEFINITE MATRIX PAIRS*

SUZANA MIODRAGOVIĆ[†], NINOSLAV TRUHAR[†], AND IVANA KUZMANOVIĆ IVIČIĆ[†]

Abstract. In this paper, we consider perturbations of a Hermitian matrix pair (H, M) , where $H = GJG^*$ is non-singular, $J = \text{diag}(\pm 1)$, and M is a positive definite matrix. The corresponding perturbed pair defined as $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ is such that $\tilde{H} = \tilde{G}J\tilde{G}^*$ is non-singular and \tilde{M} is a positive definite matrix. An upper bound for the norm of the tangents of the angles between the eigenspaces of the perturbed and unperturbed pairs is derived. The rotation of the eigenspaces under a perturbation is measured in the scalar product induced by M . We show that a relative $\tan \Theta$ -bound for the standard eigenvalue problem is a special case of our new bound.

Key words. perturbation of matrix pairs, rotation of subspaces, tangent theta theorem, eigenvalues, eigenspaces

AMS subject classifications. 65F99, 15A42, 65F15, 47A55

1. Introduction and motivation. The $\tan \Theta$ -theorem is one of the main theorems, among the other theorems given in [2], that is used to examine the quality of approximated eigenspaces for the standard eigenvalue problem.

In general, perturbation theory for the eigenspace of the standard, general, and quadratic eigenvalue problem has developed into two major branches: the so-called absolute (see [2, 11, 15, 19]) and relative perturbation theory (see [1, 3, 5, 6, 7, 16]). Absolute bounds for invariant subspaces bound the angle between the original and the perturbed subspace in terms of an absolute eigenvalue difference (eigenvalue separation), while relative bounds contain a relative eigenvalue separation. The estimates derived from absolute and relative bounds can be very different. The choice between these bounds depends on the specific matrix and the perturbation under consideration. There is no general consensus on which type of bound yields the most accurate results for a given matrix and perturbation. One advantage of relative perturbation bounds is that they are better at exploiting structures in the perturbations than absolute bounds. For a more detailed discussion on the similarities and differences between relative and absolute perturbation bounds, see [9, 10].

Most perturbation bounds for eigenspaces are of the type of $\sin \Theta$ -theorems. Some extended results regarding $\tan \Theta$ -theorems, with relaxed conditions, can be found in [13, 18], and they have been recently generalized [17] motivated by a conjecture in [26, Corollary 3.22]. As it has been pointed out in [18], one important application of the $\tan \Theta$ -theorems is in examining the quality of Ritz values, which has been thoroughly studied in many papers such as [17, 25, 26].

Another recent generalization of a $\tan \Theta$ -theorem within absolute perturbation theory of definite matrix pairs can be found in [14]. Since in [14] the perturbation of one matrix has to be off-block diagonal, this may be seen as a potential drawback in applications.

The purpose of this paper is to derive $\tan \Theta$ -theorems for the matrix pair (H, M) , where H is a non-singular Hermitian matrix factorized as

$$H = GJG^*, \quad J = \text{diag}(\pm 1),$$

*Received November 2, 2023. Accepted May 3, 2024. Published online on July 2, 2024. Recommended by Froilan Dopico.

[†]School of Applied Mathematics and Informatics J. J. Strossmayer University of Osijek, Trg Ljudevita Gaja 6. HR-31000 Osijek, Croatia ({ssusic, ntruhar, ikuzmano}@mathos.hr).

where G is non-singular and M is a positive definite Hermitian matrix. The corresponding perturbed pair is $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ such that

$$\tilde{H} = \tilde{G}J\tilde{G}^*, \quad J = \text{diag}(\pm 1),$$

with \tilde{G} non-singular and \tilde{M} a positive definite Hermitian matrix. Same as in [14] and also in [6, 7], the angle operator Θ is defined in the matrix-dependent scalar product $(x, y)_M = x^*My$, where $x, y \in \mathbb{C}^n$.

As far as the authors know, previous $\tan \Theta$ -theorems for standard (see [2, 13, 18]) and generalized (see [14]) eigenvalue problems are considered as absolute perturbation bounds, derived under the assumption of eigenvalue separation.

Contrary to the above-mentioned results on $\tan \Theta$ -theorems, including the recent paper [14], the new $\tan \Theta$ -theorems in this article provide bounds considered as relative perturbation bounds. The first new bound involves very general conditions on the eigenvalue separation, but it holds only for the Frobenius norm. The second new bound requires some conditions on the eigenvalue separation, but it holds for any unitary invariant norm.

The main motivation for this paper is the absence of $\tan \Theta$ -bounds for the generalized eigenvalue problem in the literature, even though they exist for the standard eigenvalue problem. The advantage of a tangent angle bound over a sine angle bound follows from the fact that for sharp angles, $0 \leq \theta < \frac{\pi}{2}$, the inequality $\tan \theta \geq \sin \theta$ holds, as pointed out in [18]. Recently derived relative $\sin \Theta$ -bounds given in [6] are compared with the new bounds for $\tan \Theta$, and it is shown that, under the assumption that $0 \leq \theta < \frac{\pi}{3}$, i.e., $\cos \theta < 1/2$, these two bounds are almost equally sharp. This also follows from the fact that we used similar techniques to derive the new $\tan \Theta$ -bounds.

The paper has the following structure: In Section 2, definitions and general settings are given, the most important of which is the setting of the structured Sylvester equation. From this structured equation, two main approaches for deriving our main bounds are presented in Section 3. Additionally, at the end of Section 3, two main $\tan \Theta$ -theorems are presented, one for the Frobenius norm and the other for any unitarily invariant norm. Numerical experiments are presented in Section 4.

Throughout this paper, $\|\cdot\|_2$, $\|\cdot\|_F$, and $\|\cdot\|$ stand for the spectral norm, the Frobenius norm, and any unitarily invariant norm, respectively, while $\kappa_2(A)$ stands for the condition number of matrix A in the spectral norm. Additionally, I_m (or simply I if its dimension is clear from the context) denotes the $m \times m$ identity matrix. The spectrum of a matrix A is denoted by $\lambda(A)$. The minimal and maximal singular values of A are denoted as $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$, respectively.

Throughout this paper, when referring to the eigenvalues and eigenvectors of a matrix pair, we specifically mean generalized eigenvalues and eigenvectors. However, we omit the term "generalized" when the context makes it clear that we are referring to a matrix pair.

2. Definitions and general settings. Let (H, M) be a Hermitian matrix pair with an indefinite non-singular matrix H and a positive definite matrix M of order n . Let the corresponding perturbed pair $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ also be Hermitian, with \tilde{H} being indefinite non-singular and \tilde{M} being positive definite. We consider the following generalized eigenvalue problem

$$Hx = \lambda Mx$$

and the corresponding perturbed problem

$$\tilde{H}\tilde{x} = \tilde{\lambda}\tilde{M}\tilde{x},$$

where the matrices H and \tilde{H} can be written as

$$H = GJG^* \quad \text{and} \quad \tilde{H} = \tilde{G}J\tilde{G}^*, \quad J = \text{diag}(\pm 1),$$

and where G and \tilde{G} are non-singular matrices. The matrices H and \tilde{H} are assumed, as it is common in relative perturbation theory, to have the same inertia. In the next remark we state the conditions under which this assumption is satisfied; see also [5, 22].

REMARK 2.1. Let $H = GJG^*$ be perturbed such that $\tilde{H} = G(J + E)G^*$. Under the assumption that $\|H^{-1}\|_2 \|\delta H\|_2 < 1$, it follows that $\|E\|_2 < 1$ and also $\|EJ\|_2 < 1$. Then, the following series expression from [8, Theorem 6.2.8]

$$T := (I + EJ)^{1/2} = I + \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(2i-1)!!}{2^i i!} (EJ)^i$$

obviously converge since $\|EJ\|_2 < 1$. It can be also verified that $T = JT^*J$, and therefore

$$J + E = TJT^*.$$

Now, $\|E\|_2 < 1$ implies that $H = GJG^*$ and $\tilde{H} = G(J + E)G^*$ have the same inertia as J .

Under these assumptions, the matrix pairs (H, M) and (\tilde{H}, \tilde{M}) can be simultaneously diagonalized, i.e., there exist non-singular matrices $X, \tilde{X} \in \mathbb{C}^{n \times n}$ such that

$$(2.1) \quad X^*HX = \Lambda, \quad X^*MX = I,$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{R}$, for $i = 1, \dots, n$, and

$$(2.2) \quad \tilde{X}^*\tilde{H}\tilde{X} = \tilde{\Lambda}, \quad \tilde{X}^*\tilde{M}\tilde{X} = I,$$

where $\tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$, $\tilde{\lambda}_i \in \mathbb{R}$, for $i = 1, \dots, n$; see [19, Chapter IV, Theorem 1.15].

For a given k , $1 \leq k < n$, let X and \tilde{X} be partitioned as

$$X = [X_1 \ X_2] \quad \text{and} \quad \tilde{X} = [\tilde{X}_1 \ \tilde{X}_2],$$

where $X_1, \tilde{X}_1 \in \mathbb{C}^{n \times k}$ and $X_2, \tilde{X}_2 \in \mathbb{C}^{n \times (n-k)}$. The eigenvalue decomposition (2.1) can now be written as

$$(2.3) \quad \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} H [X_1 \ X_2] = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}, \quad \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} M [X_1 \ X_2] = \begin{bmatrix} I_k & \\ & I_{n-k} \end{bmatrix},$$

where $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^{k \times k}$ and $\Lambda_2 = \text{diag}(\lambda_{k+1}, \dots, \lambda_n) \in \mathbb{R}^{(n-k) \times (n-k)}$. The same holds for the perturbed problem:

$$(2.4) \quad \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_2^* \end{bmatrix} \tilde{H} [\tilde{X}_1 \ \tilde{X}_2] = \begin{bmatrix} \tilde{\Lambda}_1 & \\ & \tilde{\Lambda}_2 \end{bmatrix}, \quad \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_2^* \end{bmatrix} \tilde{M} [\tilde{X}_1 \ \tilde{X}_2] = \begin{bmatrix} I_k & \\ & I_{n-k} \end{bmatrix},$$

where $\tilde{\Lambda}_1 = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_k) \in \mathbb{R}^{k \times k}$ and $\tilde{\Lambda}_2 = \text{diag}(\tilde{\lambda}_{k+1}, \dots, \tilde{\lambda}_n) \in \mathbb{R}^{(n-k) \times (n-k)}$. It is important to emphasize that we will use a structured Sylvester equation. Due to the uniqueness of its solution, we have to assume that the given k is such that $\Lambda(\Lambda_1) \cap \Lambda(\tilde{\Lambda}_2) = \emptyset$ and $\Lambda(\tilde{\Lambda}_1) \cap \Lambda(\Lambda_2) = \emptyset$.

Also, we have to assume that $\Lambda(\Lambda_1) \cap \Lambda(\Lambda_2) = \emptyset$ and $\Lambda(\tilde{\Lambda}_1) \cap \Lambda(\tilde{\Lambda}_2) = \emptyset$, which ensures that the subspaces $\mathcal{R}(X_1)$ spanned by the columns of X_1 and $\mathcal{R}(\tilde{X}_1)$ spanned by the columns of \tilde{X}_1 are uniquely determined.

The main goal is to give a bound for the distance between the subspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\tilde{X}_1)$, and this is achieved by bounding the tangent of the canonical angles between these subspaces. Note that the matrices X and \tilde{X} are M -orthonormal and \tilde{M} -orthonormal, respectively. Because of this, the distance between the subspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\tilde{X}_1)$ is measured using a matrix-dependent scalar product.

The following relationship between matrices that are unitary in an M - and \tilde{M} -dependent scalar product is important: for an \tilde{M} -unitary matrix \tilde{X} and a small perturbation δM , the matrix $\tilde{X}^* M \tilde{X} = I - \tilde{X}^* \delta M \tilde{X}$ is positive definite. Using the Cholesky factorization

$$(2.5) \quad \tilde{X}^* M \tilde{X} = Y Y^*,$$

where $Y \in \mathbb{C}^{n \times n}$, it follows that $Y^{-1} \tilde{X}^* M X$ is a unitary matrix because both matrices $M^{1/2} X$ and $M^{1/2} \tilde{X} Y^{-*}$ are unitary; see (2.1) and (2.5).

2.1. CS decomposition. The CS decomposition (see [12, 19]) is the main tool used to derive a tan Θ -bound. If the matrix $Y^{-1} \tilde{X}^* M X$ is partitioned such that

$$Y^{-1} \tilde{X}^* M X = \begin{bmatrix} k & n-k \\ (Y^{-1} \tilde{X}^* M X)_{11} & (Y^{-1} \tilde{X}^* M X)_{12} \\ (Y^{-1} \tilde{X}^* M X)_{21} & (Y^{-1} \tilde{X}^* M X)_{22} \end{bmatrix} \begin{matrix} k \\ n-k \end{matrix},$$

then by the CS decomposition there exist unitary matrices

$$U = \text{diag}(U_1, U_2) \in \mathbb{C}^{n \times n} \quad \text{and} \quad V = \text{diag}(V_1, V_2) \in \mathbb{C}^{n \times n},$$

with $U_1, V_1 \in \mathbb{C}^{k \times k}$ and $U_2, V_2 \in \mathbb{C}^{(n-k) \times (n-k)}$, such that

$$(2.6) \quad \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^* Y^{-1} \tilde{X}^* M X \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} C_1 & -S_1 \\ S_2 & C_2 \end{bmatrix},$$

where C_1, C_2, S_1 , and S_2 take the form

$$(2.7) \quad \begin{bmatrix} C_1 & -S_1 \\ S_2 & C_2 \end{bmatrix} = \begin{cases} \left[\begin{array}{c|cc} C & 0 & -S \\ \hline 0 & I_{n-2k} & 0 \\ S & 0 & C \end{array} \right], & \text{for } k < \frac{n}{2}, \\ \left[\begin{array}{cc|c} C & -S & \\ \hline S & C & \end{array} \right], & \text{for } k = \frac{n}{2}, \\ \left[\begin{array}{cc|c} I_{2k-n} & 0 & 0 \\ \hline 0 & C & -S \\ 0 & S & C \end{array} \right], & \text{for } k > \frac{n}{2}. \end{cases}$$

In (2.7), the matrices C and S are given by

$$C = \text{diag}(\cos \theta_1, \dots, \cos \theta_p) \quad \text{and} \quad S = \text{diag}(\sin \theta_1, \dots, \sin \theta_p),$$

where $\theta_i \in [0, \pi/2)$, for $i = 1, \dots, p$, $p = \min\{k, n-k\}$, are the canonical angles between the eigenspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\tilde{X}_1)$ given in the M -dependent scalar product (for more details, see [7, Section 2.2], [12, Section 4]). The trigonometric functions of the angle between the eigenspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\tilde{X}_1)$ in the M -dependent scalar product are the same as the

trigonometric functions of the angle between the eigenspaces $\mathcal{R}(M^{1/2}X_1)$ and $\mathcal{R}(M^{1/2}\tilde{X}_1)$ in the Euclidean scalar product. Also, note that

$$SC^{-1} = \text{diag}(\tan \theta_1, \dots, \tan \theta_p).$$

REMARK 2.2. Note that $[C_1^{-1}]_{ii} \geq 1$ and $[C_2^{-1}]_{ii} \geq 1$ in all three cases from (2.7), which means that $\|C_1^{-1}\|_2 = \|C_2^{-1}\|_2 = \|C^{-1}\|_2$. Also, in all three cases, $\|S_1C_2^{-1}\| = \|SC^{-1}\|$.

Let Y be partitioned as $\begin{bmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{bmatrix}$ with a $k \times k$ matrix Y_{11} . Then,

$$Y^{-1} = \begin{bmatrix} Y_{11}^{-1} & 0 \\ -Y_{22}^{-1}Y_{21}Y_{11}^{-1} & Y_{22}^{-1} \end{bmatrix}.$$

Further, from (2.6), we obtain

$$(2.8) \quad \begin{bmatrix} Y_{11}^{-1}\tilde{X}_1^*MX_1 & Y_{11}^{-1}\tilde{X}_1^*MX_2 \\ -Y_{22}^{-1}Y_{21}Y_{11}^{-1}\tilde{X}_1^*MX_1 + Y_{22}^{-1}\tilde{X}_2^*MX_1 & -Y_{22}^{-1}Y_{21}Y_{11}^{-1}\tilde{X}_1^*MX_2 + Y_{22}^{-1}\tilde{X}_2^*MX_2 \end{bmatrix} = U \begin{bmatrix} C_1 & -S_1 \\ S_2 & C_2 \end{bmatrix} V^*.$$

Now the basic concepts for obtaining the main equation are set.

2.2. Setting the main equation. From (2.3), it is easy to see that

$$HX_2 = MX_2\Lambda_2,$$

and by multiplying from the left with \tilde{X}_1^* , we get

$$\tilde{X}_1^*HX_2 = \tilde{X}_1^*MX_2\Lambda_2.$$

Now, using that $\tilde{H} = H + \delta H$ and $\tilde{X}_1^*\tilde{H} = \tilde{\Lambda}_1\tilde{X}_1^*\tilde{M}$, we have

$$\tilde{\Lambda}_1\tilde{X}_1^*\tilde{M}X_2 - \tilde{X}_1^*\delta HX_2 = \tilde{X}_1^*MX_2\Lambda_2,$$

which, after using $\tilde{M} = M + \delta M$, also yields

$$\tilde{\Lambda}_1Y_{11}Y_{11}^{-1}\tilde{X}_1^*MX_2 + \tilde{\Lambda}_1\tilde{X}_1^*\delta MX_2 - \tilde{X}_1^*\delta HX_2 = Y_{11}Y_{11}^{-1}\tilde{X}_1^*MX_2\Lambda_2.$$

Further, the equality $U_1S_1V_2^* = -Y_{11}^{-1}\tilde{X}_1^*MX_2$ from (2.8) gives

$$(2.9) \quad -\tilde{\Lambda}_1Y_{11}U_1S_1V_2^* = -Y_{11}U_1S_1V_2^*\Lambda_2 + \tilde{X}_1^*\delta HX_2 - \tilde{\Lambda}_1\tilde{X}_1^*\delta MX_2.$$

Multiplying (2.9) from the right by $V_2C_2^{-1}$, we obtain

$$(2.10) \quad -\tilde{\Lambda}_1Y_{11}U_1S_1C_2^{-1} = -Y_{11}U_1S_1V_2^*\Lambda_2V_2C_2^{-1} + \tilde{X}_1^*\delta HX_2V_2C_2^{-1} - \tilde{\Lambda}_1\tilde{X}_1^*\delta MX_2V_2C_2^{-1}.$$

By reordering the above equation, we get

$$(2.11) \quad -\tilde{\Lambda}_1Y_{11}U_1S_1C_2^{-1} + Y_{11}U_1S_1V_2^*\Lambda_2V_2C_2^{-1} = \tilde{X}_1^*\delta HX_2V_2C_2^{-1} - \tilde{\Lambda}_1\tilde{X}_1^*\delta MX_2V_2C_2^{-1}.$$

3. The tan Θ -bound—the main result. In this section, we derive upper bounds for the norm of the tangent of the angle between the subspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\tilde{X}_1)$, that is, for

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\| = \|SC^{-1}\|,$$

where Θ is measured in a matrix-dependent scalar product. This is done in two ways. The first bound is obtained as a direct solution of the previously derived equation (2.11). It holds only for the Frobenius norm without any additional conditions on the spectrum (except on a separation condition). The second bound is derived for any unitarily invariant norm. It is also obtained from equation (2.10) using standard linear algebra techniques. However, unlike the previous bound, this bound requires some additional conditions on the spectrum. Now we state the first theorem of this paper.

THEOREM 3.1. *Let (H, M) and $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be Hermitian matrix pairs, where H and \tilde{H} are non-singular and M and \tilde{M} are positive definite matrices of order n . Let $X = [X_1 \ X_2]$ and $\tilde{X} = [\tilde{X}_1 \ \tilde{X}_2]$ be the non-singular matrices from (2.3) and (2.4) that simultaneously diagonalize the pairs (H, M) and (\tilde{H}, \tilde{M}) , respectively. Then the following bound holds:*

$$(3.1) \quad \|SC^{-1}\|_F \leq \frac{\|C_2^{-1}\|_2^2 \cdot \|Y_{11}^{-1}\|_2 \cdot \left(\|\tilde{X}_1^* \delta H X_2\|_F + \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\|_F \right)}{\min_{\substack{i=1, \dots, k \\ j=k+1, \dots, n}} |\tilde{\lambda}_i - \lambda_j|}.$$

Proof. If we set $\Psi := Y_{11} U_1 S_1 C_2^{-1}$ in (2.11), it is easy to see that (2.11) is a standard Sylvester equation

$$\tilde{\Lambda}_1 \Psi - \Psi C_2 V_2^* \Lambda_2 V_2 C_2^{-1} = \tilde{X}_1^* \delta H X_2 V_2 C_2^{-1} - \tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1},$$

where the bound for the Frobenius norm of the solution, i.e., $\|\Psi\|_F$, is given as (see [4, 19])

$$(3.2) \quad \|\Psi\|_F \leq \frac{\|\tilde{X}_1^* \delta H X_2 V_2 C_2^{-1}\|_F + \|\tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1}\|_F}{\text{sep}(\tilde{\Lambda}_1, C_2 V_2^* \Lambda_2 V_2 C_2^{-1})}.$$

The denominator in (3.2) is the separation of $\tilde{\Lambda}_1$ and $C_2 V_2^* \Lambda_2 V_2 C_2^{-1}$. By [19, p. 245], for this separation we have

$$\text{sep}(\tilde{\Lambda}_1, C_2 V_2^* \Lambda_2 V_2 C_2^{-1}) \geq \frac{|\Lambda(\tilde{\Lambda}_1) - \Lambda(C_2 V_2^* \Lambda_2 V_2 C_2^{-1})|}{\kappa_2(Y_a) \kappa_2(Y_b)},$$

where Y_a and Y_b are matrices of the eigenvectors of $\tilde{\Lambda}_1$ and $C_2 V_2^* \Lambda_2 V_2 C_2^{-1}$. Note that $\Lambda(C_2 V_2^* \Lambda_2 V_2 C_2^{-1}) = \Lambda(\Lambda_2)$ since the matrices $C_2 V_2^* \Lambda_2 V_2 C_2^{-1}$ and Λ_2 are similar. Also, note that $\kappa_2(Y_a) = 1$ and that $\kappa_2(Y_b) \leq \|C_2^{-1}\|_2$. This means that for the separation we have

$$(3.3) \quad \begin{aligned} \text{sep}(\tilde{\Lambda}_1, C_2 V_2^* \Lambda_2 V_2 C_2^{-1}) &\geq \frac{1}{\|C_2^{-1}\|_2} |\Lambda(\tilde{\Lambda}_1) - \Lambda(\Lambda_2)| \\ &\geq \frac{1}{\|C_2^{-1}\|_2} \min_{\substack{i=1, \dots, k \\ j=k+1, \dots, n}} |\tilde{\lambda}_i - \lambda_j|. \end{aligned}$$

Additionally, it holds

$$(3.4) \quad \begin{aligned} \|\tilde{X}_1^* \delta H X_2 V_2 C_2^{-1}\|_F &\leq \|C_2^{-1}\|_2 \|\tilde{X}_1^* \delta H X_2\|_F, \\ \|\tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1}\|_F &\leq \|C_2^{-1}\|_2 \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\|_F \end{aligned}$$

and also

$$(3.5) \quad \|\Psi\|_F = \|Y_{11}U_1S_1C_2^{-1}\|_F \geq \sigma_{\min}(Y_{11})\|SC^{-1}\|_F = \frac{1}{\|Y_{11}^{-1}\|_2}\|SC^{-1}\|_F,$$

where $\|S_1C_2^{-1}\|_F = \|SC^{-1}\|_F$; see Remark 2.2. Inserting (3.3), (3.4), and (3.5) into (3.2), we obtain the bound (3.1). \square

The following lemma is an important tool that we use in the proof of the next theorem.

LEMMA 3.2 ([18, Lemma 2.1]). *Let $X \in \mathbb{C}^{m \times n}$, $Y \in \mathbb{C}^{n \times r}$, and $Z \in \mathbb{C}^{r \times s}$ have singular value decompositions $X = U_X \Sigma_X V_X^*$, $Y = U_Y \Sigma_Y V_Y^*$, and $Z = U_Z \Sigma_Z V_Z^*$, where the singular values are arranged in descending order. Then, for any unitarily invariant norm $\|\cdot\|$,*

$$\begin{aligned} \|XYZ\| &\leq \|Y\|_2 \|\tilde{\Sigma}_X \tilde{\Sigma}_Z\|, \\ \|XYZ\| &\leq \|X\|_2 \|\tilde{\Sigma}_Y \tilde{\Sigma}_Z\|, \\ \|XYZ\| &\leq \|Z\|_2 \|\tilde{\Sigma}_X \tilde{\Sigma}_Y\|, \end{aligned}$$

where $\tilde{\Sigma}_X$, $\tilde{\Sigma}_Y$, and $\tilde{\Sigma}_Z$ are diagonal matrices of the p largest singular values and $p = \min\{m, n, r, s\}$.

Now we state the second result that gives a bound that holds for any unitarily invariant norm.

THEOREM 3.3. *Let (H, M) and $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be Hermitian matrix pairs, where H and \tilde{H} are non-singular and M and \tilde{M} are positive definite matrices of order n . Let $X = [X_1 \ X_2]$ and $\tilde{X} = [\tilde{X}_1 \ \tilde{X}_2]$ be the non-singular matrices from (2.3) and (2.4) that simultaneously diagonalize the pairs (H, M) and (\tilde{H}, \tilde{M}) , respectively. Then, if $\sigma_{\min}(\tilde{\Lambda}_1) - \kappa(Y_{11})\|\Lambda_2\|_2 > 0$, the following bound holds:*

$$(3.6) \quad \|SC^{-1}\| \leq \frac{\|C_2^{-1}\|_2 \cdot \left(\|\tilde{X}_1^* \delta H X_2\| + \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\| \right) \cdot \|Y_{11}^{-1}\|_2}{\sigma_{\min}(\tilde{\Lambda}_1) - \kappa(Y_{11})\|\Lambda_2\|_2}.$$

Proof. We start this proof with the equation (2.10). By taking any unitarily invariant norm of equation (2.10), we get

$$\begin{aligned} \|\tilde{\Lambda}_1 Y_{11} U_1 S_1 C_2^{-1}\| &\leq \|Y_{11} U_1 S_1 V_2^* \Lambda_2 V_2 C_2^{-1}\| + \|\tilde{X}_1^* \delta H X_2 V_2 C_2^{-1}\| \\ &\quad + \|\tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1}\|. \end{aligned}$$

Also

$$(3.7) \quad \begin{aligned} \|\tilde{\Lambda}_1 Y_{11} U_1 S_1 C_2^{-1}\| - \|Y_{11} U_1 S_1 V_2^* \Lambda_2 V_2 C_2^{-1}\| \\ \leq \|\tilde{X}_1^* \delta H X_2 V_2 C_2^{-1}\| + \|\tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1}\|. \end{aligned}$$

Note that

$$(3.8) \quad \|\tilde{\Lambda}_1 Y_{11} U_1 S_1 C_2^{-1}\| \geq \sigma_{\min}(\tilde{\Lambda}_1) \sigma_{\min}(Y_{11}) \cdot \|S_1 C_2^{-1}\|$$

and

$$(3.9) \quad \|\tilde{X}_1^* \delta H X_2 V_2 C_2^{-1}\| \leq \|\tilde{X}_1^* \delta H X_2\| \cdot \|C_2^{-1}\|_2.$$

In addition,

$$(3.10) \quad \|\tilde{\Lambda}_1 \tilde{X}_1^* \delta M X_2 V_2 C_2^{-1}\| \leq \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\| \|C_2^{-1}\|_2,$$

and finally, from Lemma 3.2, we obtain

$$(3.11) \quad \|Y_{11} U_1 S_1 V_2^* \Lambda_2 V_2 C_2^{-1}\| \leq \|Y_{11}\|_2 \|\Lambda_2\|_2 \|\tilde{\Sigma}_{S_1} \tilde{\Sigma}_{C_2^{-1}}\|,$$

where

$$\begin{aligned} \tilde{\Sigma}_{S_1} &= \text{diag}(\sin \theta_p, \dots, \sin \theta_1), \\ \tilde{\Sigma}_{C_2} &= \text{diag}(\cos \theta_1, \dots, \cos \theta_p), \\ \tilde{\Sigma}_{C_2^{-1}} &= \text{diag}\left(\frac{1}{\cos \theta_p}, \dots, \frac{1}{\cos \theta_1}\right), \end{aligned}$$

$\theta_1 \leq \theta_2 \leq \dots \leq \theta_p$, and $p = \min\{k, n - k\}$. Thus,

$$\|\tilde{\Sigma}_{S_1} \tilde{\Sigma}_{C_2^{-1}}\| = \|\text{diag}(\tan \theta_p, \dots, \tan \theta_1)\| = \|S_1 C_2^{-1}\| = \|S C^{-1}\|$$

(see Remark 2.2). Now, from (3.7) using (3.8), (3.9), (3.10), and (3.11), we obtain

$$\begin{aligned} \|S C^{-1}\| \left(\sigma_{\min}(\tilde{\Lambda}_1) \sigma_{\min}(Y_{11}) - \|Y_{11}\|_2 \|\Lambda_2\|_2 \right) \\ \leq \|C_2^{-1}\|_2 \cdot \left(\|\tilde{X}_1^* \delta H X_2\| + \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\| \right). \end{aligned}$$

Using $\sigma_{\min}(Y_{11}) = \frac{1}{\|Y_{11}^{-1}\|_2}$ and the assumption that $\sigma_{\min}(\tilde{\Lambda}_1) \sigma_{\min}(Y_{11}) - \|Y_{11}\|_2 \|\Lambda_2\|_2 \geq 0$, we derive the inequality

$$\|S C^{-1}\| \leq \frac{\|C_2^{-1}\|_2 \cdot \left(\|\tilde{X}_1^* \delta H X_2\| + \|\tilde{\Lambda}_1\|_2 \|\tilde{X}_1^* \delta M X_2\| \right) \cdot \|Y_{11}^{-1}\|_2}{\sigma_{\min}(\tilde{\Lambda}_1) - \kappa_2(Y_{11}) \|\Lambda_2\|_2}. \quad \square$$

REMARK 3.4. Assume that the $\tan \theta$ -bound is derived only for small perturbations of the matrix pair such that the angle θ between the eigenvectors is $0 \leq \theta \leq \frac{\pi}{3}$. Under this assumption, the appearance of the expression $\|C_2^{-1}\|_2$, which is related to the cosine of the angles, can be removed from (3.6). More precisely, it follows that

$$\|C_2^{-1}\|_2 = \frac{1}{\cos \theta_k} \leq 2,$$

when $\theta_k \leq \frac{\pi}{3}$ and $\theta_1 < \theta_2 < \dots < \theta_k$.

Note that the bound (3.1) from Theorem 3.1 and also the bound (3.6) from Theorem 3.3 require too much information about the subspaces (perturbed and unperturbed). Therefore, in the following, we will simplify the right-hand side of (3.1) and (3.6) by deriving bounds for $\|\tilde{X}_1^* \delta H X_2\|$, $\|\tilde{X}_1^* \delta M X_2\|$, $\|Y_{11}^{-1}\|_2$, and $\kappa_2(Y_{11})$. This will give the main results of this paper.

The eigenvalue problems for $H = G J G^*$ and $\tilde{H} = \tilde{G} J \tilde{G}^*$ are closely related to the hyperbolic eigenvalue problem for the pair $(G^* G, J)$ and $(\tilde{G}^* \tilde{G}, J)$ (see, e.g., [20]). From [19, Theorem VI.1.15] and [19, Corollary VI.1.19] it follows that there always exist J -unitary matrices F and \tilde{F} , that is, $F^* J F = J$ and $\tilde{F}^* J \tilde{F} = J$, that simultaneously diagonalize the pairs $(G^* G, J)$ and $(\tilde{G}^* \tilde{G}, J)$, respectively. More about J -unitary matrices and their

properties can be found in [23]. Since $H = GJG^*$ and $X^*GJG^*X = \Lambda$, by spectral calculus one can analyze

$$(3.12) \quad G^*X = F|\Lambda|^{1/2}.$$

Similarly for the perturbed quantities we have $\tilde{H} = \tilde{G}\tilde{J}\tilde{G}^*$ and $\tilde{X}^*\tilde{G}\tilde{J}\tilde{G}^*\tilde{X} = \tilde{\Lambda}$ and

$$(3.13) \quad \tilde{G}^*\tilde{X} = \tilde{F}|\tilde{\Lambda}|^{1/2}.$$

Now,

$$\tilde{X}_1^*\delta H X_2 = \tilde{X}_1^*\tilde{G}\tilde{G}^{-1}\delta H G^{-*}G^*X_2$$

together with (3.12) and (3.13) gives

$$(3.14) \quad \tilde{X}_1^*\delta H X_2 = |\tilde{\Lambda}_1|^{1/2}\tilde{F}_1^*\tilde{G}^{-1}\delta H G^{-*}F_2|\Lambda_2|^{1/2}.$$

It is assumed here that $F = [F_1 \ F_2]$ and $\tilde{F} = [\tilde{F}_1 \ \tilde{F}_2]$ are partitioned accordingly to (2.3).

Now, from (3.14) we can see that

$$(3.15) \quad \|\tilde{X}_1^*\delta H X_2\| \leq \sigma_{\max}(|\tilde{\Lambda}_1|^{1/2}) \cdot \sigma_{\max}(|\Lambda_2|^{1/2})\|\tilde{F}\|_2\|F\|_2\|\tilde{G}^{-1}\delta H G^{-*}\|.$$

Also, since

$$\tilde{X}_1^*\delta M X_2 = \tilde{X}_1^*\tilde{M}^{1/2}\tilde{M}^{-1/2}\delta M M^{-1/2}M^{1/2}X_2$$

and the columns of the matrices $\tilde{Q}_1 := \tilde{X}_1^*\tilde{M}^{1/2}$ and $Q_2 := M^{1/2}X_2$ are orthonormal, we obtain

$$(3.16) \quad \|\tilde{X}_1^*\delta M X_2\| = \|Q_1^*\tilde{M}^{-1/2}\delta M M^{-1/2}Q_2\| = \|\tilde{M}^{-1/2}\delta M M^{-1/2}\|.$$

Under the assumption that

$$\eta_M := \|M^{-1/2}\delta M M^{-1/2}\| < \frac{1}{2},$$

in the proof of [6, Theorem 3.4] the following bound is given for $\|Y_{11}^{-1}\|_2$:

$$(3.17) \quad \|Y_{11}^{-1}\|_2 \leq \frac{\sqrt{1 - \eta_M}}{\sqrt{1 - 2\eta_M}}.$$

Also, we can bound $\kappa_2(Y_{11})$ by bounding $\|Y_{11}\|_2 = \|\sqrt{I - \tilde{X}_1^*\delta M \tilde{X}_1}\|_2$, where on the right-hand side we have the norm of the Cholesky factor of $I - \tilde{X}_1^*\delta M \tilde{X}_1$. Note that

$$(3.18) \quad \|Y_{11}\|_2 \leq \sqrt{1 + \|\tilde{X}_1^*\delta M \tilde{X}_1\|_2} \leq \sqrt{1 + \|\tilde{X}^*\delta M \tilde{X}\|_2}.$$

Using the \tilde{M} -unitarity of \tilde{X} and $M^{-1/2}(I + M^{-1/2}\delta M M^{-1/2})^{-1/2}$, we can write

$$(3.19) \quad \tilde{X} = M^{-1/2}(I + M^{-1/2}\delta M M^{-1/2})^{-1/2}Q,$$

where Q is a unitary matrix. If we set $W = M^{-1/2}\delta M M^{-1/2}$, then from (3.19) it follows

$$(3.20) \quad \|\tilde{X}^*\delta M \tilde{X}\|_2 = \|(I + W)^{-1/2}W(I + W)^{-1/2}\|_2 \leq \frac{\eta_M}{1 - \eta_M}.$$

Inserting (3.20) into (3.18) we obtain

$$(3.21) \quad \|Y_{11}\|_2 \leq \frac{1}{\sqrt{1-\eta_M}}.$$

From (3.17) and (3.21) it follows

$$(3.22) \quad \kappa_2(Y_{11}) \leq \frac{1}{\sqrt{1-2\eta_M}} =: \alpha_M.$$

Finally, we can state the two main theorems of this paper. Recall that Θ is measured in a matrix-dependent scalar product.

THEOREM 3.5. *Let the same assumptions as in Theorem 3.3 hold. Let H and \tilde{H} be of the form $H = GJG^*$ and $\tilde{H} = \tilde{G}J\tilde{G}^*$, where G and \tilde{G} are non-singular matrices. Further, let F and \tilde{F} be J -unitary matrices that simultaneously diagonalize the pairs (G^*G, J) and $(\tilde{G}^*\tilde{G}, \tilde{J})$, respectively. Then, if $\eta_M := \|M^{-1/2}\delta MM^{-1/2}\| < \frac{1}{2}$ and $\theta \leq \frac{\pi}{3}$, the following bound holds:*

$$(3.23) \quad \begin{aligned} & \|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \\ & \leq 4 \cdot \frac{\sqrt{1-\eta_M}}{\sqrt{1-2\eta_M}} \cdot \left(\frac{\|\tilde{F}\|_2 \|F\|_2 \|\tilde{G}^{-1}\delta HG^{-*}\|_F}{\mathfrak{S}_1} + \frac{\|\tilde{M}^{-1/2}\delta MM^{-1/2}\|_F}{\mathfrak{S}_2} \right), \end{aligned}$$

where

$$\mathfrak{S}_1 = \frac{\min_{\substack{i=1,\dots,k \\ j=k+1,\dots,n}} |\tilde{\lambda}_i - \lambda_j|}{\sigma_{\max}(|\tilde{\Lambda}_1|^{1/2}) \cdot \sigma_{\max}(|\Lambda_2|^{1/2})} \quad \text{and} \quad \mathfrak{S}_2 = \frac{\min_{\substack{i=1,\dots,k \\ j=k+1,\dots,n}} |\tilde{\lambda}_i - \lambda_j|}{\|\tilde{\Lambda}_1\|_2}.$$

Proof. Using (3.15), (3.16), and (3.17) in the bound (3.1), from Theorem 3.1 we obtain the bound (3.23). \square

THEOREM 3.6. *Let the same assumptions as in Theorem 3.5 hold. If*

$$\sigma_{\min}(\tilde{\Lambda}_1) - \frac{1}{\sqrt{1-2\eta_M}} \|\Lambda_2\|_2 > 0,$$

then the following bound holds:

$$(3.24) \quad \begin{aligned} & \|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\| \\ & \leq 2 \cdot \frac{\sqrt{1-\eta_M}}{\sqrt{1-2\eta_M}} \cdot \left(\frac{\|\tilde{F}\|_2 \|F\|_2 \|\tilde{G}^{-1}\delta HG^{-*}\|}{\mathfrak{G}_1} + \frac{\|\tilde{M}^{-1/2}\delta MM^{-1/2}\|}{\mathfrak{G}_2} \right), \end{aligned}$$

where

$$\mathfrak{G}_1 = \frac{\sigma_{\min}(\tilde{\Lambda}_1) - \alpha_M \|\Lambda_2\|_2}{\sigma_{\max}(|\tilde{\Lambda}_1|^{1/2}) \cdot \sigma_{\max}(|\Lambda_2|^{1/2})} \quad \text{and} \quad \mathfrak{G}_2 = \frac{\sigma_{\min}(\tilde{\Lambda}_1) - \alpha_M \|\Lambda_2\|_2}{\|\tilde{\Lambda}_1\|_2}.$$

Proof. Using (3.15), (3.16), (3.17), and (3.22) in the bound (3.6), from Theorem 3.3 we obtain the bound (3.24). \square

REMARK 3.7. Note that the assumption $\sigma_{\min}(\tilde{\Lambda}_1) - \frac{1}{\sqrt{1-2\eta_M}} \|\Lambda_2\|_2 > 0$ in Theorem 3.6 ensures a positive denominator in (3.6), from which we derived (3.24), and this represents a condition on the separation between $|\Lambda_2|$ and $|\tilde{\Lambda}_1|$.

In [6] the authors derived a bound for the norm of the matrices F and \tilde{F} from (3.12) and (3.13) that appear in the bounds (3.23) and (3.24). More details about this can be found therein.

REMARK 3.8. Note that the values η_M and α_M are dependent on the perturbation of the matrix M . If $\delta M = 0$, then $\eta_M = 0$ and $\alpha_M = 1$, in which case the bounds (3.23) and (3.24) are of the form

$$(3.25) \quad \|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_{\text{ui}} \leq \frac{\|\tilde{F}\|_2 \|F\|_2 \|\tilde{G}^{-1} \delta H G^{-*}\|_{\text{ui}}}{\mathcal{D}},$$

where $\mathcal{D} = S_1/4$ when $\text{ui} = F$ or $\mathcal{D} = \mathcal{G}_1/2$ when ui is any unitarily invariant norm. Additionally, if H is positive definite, then \tilde{F} and F are unitary, and the bound (3.25) has the form

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_{\text{ui}} \leq \frac{\|\tilde{G}^{-1} \delta H G^{-*}\|_{\text{ui}}}{\mathcal{D}}.$$

4. Performances of the bounds. In this section, we illustrate the performance of our bounds (3.23) and (3.24). We compare them with the exact values as well as with some known perturbation bounds. More precisely, we compare the bounds (3.23) and (3.24) with the $\sin \Theta$ -bound from [6] as well as with the bound from [21]. This comparison makes sense because we consider only small perturbations of the eigenvectors for which the values of the tangent and the sines of the angle are extremely close. The $\tan \Theta$ -bound for the standard eigenvalue problem from [18] is comparable to the special cases of our bounds (3.23) and (3.24) with $M = I$ and $\delta M = 0$; see Remark 3.8.

4.1. Example 1. We consider the generalized eigenvalue problem

$$Hx = \lambda Mx,$$

with $H = \text{gallery}('ris', n) - 0.002 * \text{diag}(2 * n : -2 : 1)$, where 'ris' is the matrix from MATLAB's matrix set and $M = \text{diag}(1 : n)$ with $n = 50$. We compare the bounds (3.23) and (3.24) in the spectral and Frobenius norm with the exact values of

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \quad \text{and} \quad \|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_2.$$

We consider 30 random perturbations δH and δM of the same size, which satisfy

$$|(\delta H)_{ij}| \leq 10^{-7}, \quad |(\delta M)_{ij}| \leq 10^{-7}.$$

In the results illustrated in Figure 4.1, we can see that the bound (3.24) in the 2-norm is the sharpest, as expected. Comparing the bounds (3.24) and (3.23) in the Frobenius norm, we can notice that these bounds are equally sharp. This behavior is also expected from a theoretical point of view. The drawback of (3.24) is the strong assumption on the spectrum, which makes it inapplicable in some examples where this assumption does not hold.

In Figure 4.1, we also observe that the bounds in the Frobenius and in the spectral norms behave similarly for different perturbations of the same size.

4.2. Example 2. In this example, we compare the bounds (3.23) and (3.24) with the $\sin \Theta$ -bounds from [6] (denoted by GTM) and [21] (denoted by Sun), which are stated in the following theorems:

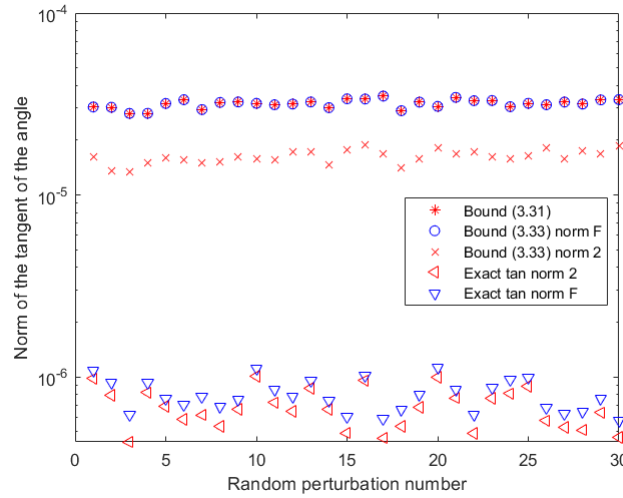


FIG. 4.1. Comparison of the bounds (3.23) and (3.24) with the exact values.

THEOREM 4.1 ([6, Theorem 3.4]). *Let (H, M) be a Hermitian pair, and let the matrices $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be the perturbed pair. Let $X = [X_1 \ X_2]$ and $\tilde{X} = [\tilde{X}_1 \ \tilde{X}_2]$ be non-singular matrices that simultaneously diagonalize the pairs (H, M) and (\tilde{H}, \tilde{M}) as in (2.1) and (2.2), respectively. If $\eta_M = \|M^{-1/2} \delta M M^{-1/2}\|_2 < \frac{1}{2}$, then*

$$(4.1) \quad \begin{aligned} & \|\sin \Theta_M(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \\ & \leq \frac{\|F\|_2 \|\tilde{F}\|_2}{\text{RelGap}(\hat{\Lambda}_1, \Lambda_2)} \cdot \Psi_H + \frac{1}{\text{RGap}(\tilde{\Lambda}_1, \hat{\Lambda}_2)} \cdot \frac{\sqrt{1 - \eta_M}}{\sqrt{1 - 2\eta_M}} \cdot \Psi_M, \end{aligned}$$

where $\Psi_H = \|G^{-1} \delta H \tilde{G}^{-*}\|_F$ and $\Psi_M = \|M^{-1/2} \delta M \tilde{M}^{-1/2}\|_F$, and

$$\begin{aligned} \text{RelGap}(\hat{\Lambda}_1, \Lambda_2) &= \min_{\substack{i=k+1, \dots, n \\ j=1, \dots, k}} \frac{|\lambda_i - \hat{\lambda}_j|}{\sqrt{|\lambda_i| |\hat{\lambda}_j|}}, \\ \text{RGap}(\tilde{\Lambda}_1, \hat{\Lambda}_2) &= \min_{\substack{i=k+1, \dots, n \\ j=1, \dots, k}} \frac{|\hat{\lambda}_i - \tilde{\lambda}_j|}{|\tilde{\lambda}_j|}, \end{aligned}$$

where $\hat{\lambda}_j, j = 1, \dots, n$, are the eigenvalues of the pair (\tilde{H}, M) .

In the same paper in [6, Corollary 3.5], a similar bound for any unitarily invariant norm is also given.

THEOREM 4.2 ([21, Theorem 2.1]). *Let the definite pair (H, M) be decomposed as in (2.3), where X_1 and X_2 have orthonormal columns. Let the analogous decomposition be given for the pair $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$, and let λ_i and $\tilde{\lambda}_i, i = 1, \dots, n$, denote*

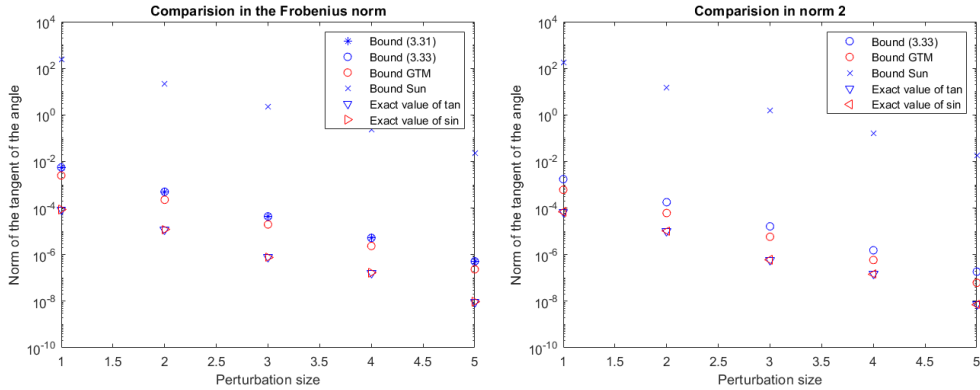


FIG. 4.2. Comparison of the bounds (3.23) and (3.24) with the $\sin \Theta$ -bounds for perturbations of different size.

the eigenvalues of (H, M) and (\tilde{H}, \tilde{M}) , respectively. If

$$\Gamma = \min \left\{ \frac{|\tilde{\lambda}_j - \lambda_i|}{\sqrt{(1 + \tilde{\lambda}_j^2)(1 + \lambda_i^2)}} : i = 1, \dots, k, j = k + 1, \dots, n \right\} > 0,$$

$$\gamma(H, M) = \min_{x \in \mathbb{C}^n, \|x\|=1} \sqrt{(x^* H x)^2 + (x^* M x)^2} > 0,$$

then

$$(4.2) \quad \|\sin \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\| \leq \frac{q \cdot \sqrt{\|H^2 + M^2\|_2} \sqrt{\|\delta H X_1\|^2 + \|\delta M X_1\|^2}}{\gamma(H, M) \gamma(\tilde{H}, \tilde{M}) \Gamma},$$

where the constant q satisfies $1 \leq q \leq \sqrt{2}$, $\gamma(H, M)$ is the Crawford number of the pair (H, M) (see, e.g., [19, Theorem 3.9, Chapter VI]), and Γ is an absolute measure of the gap in the spectrum.

EXPERIMENT 1. The first comparison is made for the example of the generalized eigenvalue problem $Hx = \lambda Mx$ with

$$H = \text{gallery}('ris', n) - 0.005 * \text{diag}(2 * n : -2 : 1),$$

where 'ris' is the matrix from MATLAB's matrix set and

$$M = 0.03 * \text{diag}(1 : 0.1 : 0.1 * (n + 9))$$

with $n = 50$ and $k = 3$. The perturbations δH and δM are of the same size and satisfy

$$|(\delta H)_{ij}| \leq \epsilon, \quad |(\delta M)_{ij}| \leq \epsilon,$$

where ϵ is gradually changing from 10^{-9} to 10^{-5} . The obtained results for the Frobenius (left) and the 2-norm (right) are presented in Figure 4.2.

EXPERIMENT 2. In this experiment, a comparison is done with the parameter-dependent matrix H of the form

$$H = \text{gallery}('ris', n) - v * \text{diag}(2 * n : -2 : 1),$$

with $v = 0.1 : -0.01 : 0.01$, $M = 0.03 * \text{diag}(1 : 0.1 : 0.1 * (n + 9))$, $n = 50$, and $k = 3$. The perturbations satisfy $|(\delta H)_{ij}| \leq 10^{-8}$, and $|(\delta M)_{ij}| \leq 10^{-8}$. The obtained results are presented in Figure 4.3.

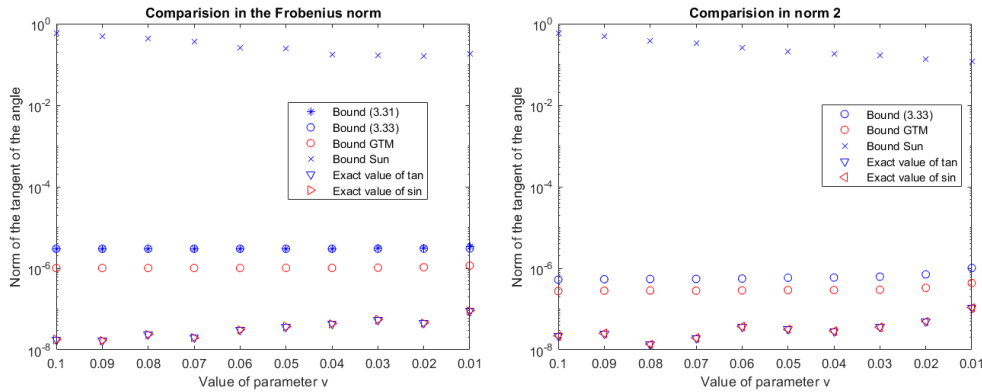


FIG. 4.3. Comparison of the bounds (3.23) and (3.24) with the $\sin \Theta$ -bounds for different values of the parameter v in the matrix H .

From the Experiments 1 and 2 in this example, we can conclude that the bounds (3.24) and (3.23) in the Frobenius norm and also in the 2-norm are almost equally sharp as (4.1) and significantly sharper than the bound (4.2) in the appropriate norms.

EXPERIMENT 3. In this experiment, we consider $H = \text{gallery}('ris', n)$ and $M = \text{diag}(1:n)$, $n = 50$, and random perturbations δH and δM satisfying

$$|(\delta H)_{ij}| \leq 10^{-8} \quad \text{and} \quad |(\delta M)_{ij}| \leq 10^{-8}.$$

For this example, we choose a subspace such that the condition from Theorem 3.6 is not satisfied, and therefore the bound (3.24) cannot be applied. Therefore, we calculate the values of the bounds (3.23) and (4.1) as well as the exact value. For $k = 2$, we obtain the exact value

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \approx 6.5496 \cdot 10^{-8}.$$

The bound (4.1) gives

$$\|\sin \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \leq 1.4013 \cdot 10^{-6},$$

while (3.23) gives

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \leq 3.6403 \cdot 10^{-6}.$$

This example shows that (3.23) can be of the same order of magnitude as (4.1), which is particularly important in cases like this where the bound (3.24) cannot be applied.

4.3. Example 3. In this example, we compare a special case of (3.23) with $M = I$ and $\delta M = 0$, with the $\tan \Theta$ -bound from [18]. That bound is for the standard eigenvalue problem $Hx = \lambda x$ and is stated (using our notation) in the following theorem:

THEOREM 4.3. [18, Theorem 1] Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix, and let $X = [X_1 \ X_2]$ be its unitary eigenvector matrix such that $X^*AX = \text{diag}(\Lambda_1, \Lambda_2)$ is diagonal, where X_1 and Λ_1 have k columns. Let $\tilde{X}_1 \in \mathbb{C}^{n \times k}$ be orthogonal, and let $R = A\tilde{X}_1 - \tilde{X}_1A_1$, where $A_1 = \tilde{X}_1^*A\tilde{X}_1$. Suppose that $\lambda(\Lambda_2)$ lies in $[a, b]$ and $\lambda(A_1)$ lies in the union of $\langle -\infty, a - \delta \rangle$ and $[b + \delta, \infty)$. Then

$$(4.3) \quad \|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\| \leq \frac{\|R\|}{\delta}.$$

This example is motivated by [6, Example 5.2], where the authors constructed a so-called *abstract Bogoliubov–de Gennes*-like example. Let

$$H = \begin{bmatrix} H_{11} & 0 \\ 0 & -H_{11} \end{bmatrix}$$

be a nonsingular indefinite Hermitian matrix, and let

$$\tilde{H} = \begin{bmatrix} H_{11} & B_p \\ B_p & -H_{11} \end{bmatrix}$$

be the perturbed matrix, where H_{11} is a positive definite Hermitian matrix obtained in MATLAB as

```
n=25;
h11=[0.01:0.001:0.32, 0.5, 0.7];
[Qtmp,temp]=qr(rand(n));
H11=Qtmp*diag(h11)*Qtmp';
H11=1/2*(H11+H11');
```

and $B_p = \delta B * \text{rand}(n)$, $B_p = B_p + B'_p$, is a random perturbation with $\delta B = 10^{-5}$. Note that this perturbation is an off-diagonal block perturbation, so the latest bound from [14, Theorem 3.2] can also be applied here. However, in the case when $M = I$ and $\delta M = 0$, this bound and that of (4.3) coincide; see [14, Remark 3.5].

An illustration of the bounds is provided here for the case when the condition for the eigenvalues given in Theorem 4.3 is satisfied. Additionally, the condition from our Theorem 3.6 is satisfied as well. This means that we choose k such that the eigenvalues of $\tilde{\Lambda}_1$ lie in the union $\langle -0.8, -0.5 \rangle \cup [0.5, 0.8)$, and the eigenvalues of Λ_2 lie in $[-0.32, 0.32]$. This gives $\delta = 0.1800$, and the bound (4.3) gives

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \leq 0.0011.$$

The exact value is approximately

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \approx 3.4342 \cdot 10^{-5}.$$

The bounds (3.24) and (3.23) are equal and give

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \leq 5.5919 \cdot 10^{-5}.$$

Note that for this example, (3.24) and (3.23) are sharp and close to the exact value of the tangent, while (4.3) is not, as it contains the absolute gap δ in the denominator.

In the end, it is important to emphasize the important property of the bound (3.23). In the case when the conditions for other $\tan \Theta$ -theorems are not satisfied and therefore they are inapplicable, the bound (3.23) can still be applied and provides a good approximation of the exact value of $\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F$. For example, if we choose k , i.e., X_1 corresponding to the eigenvalues $\{-0.7, -0.012, 0.7\}$, then the exact value is approximately

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \approx 1.2923 \cdot 10^{-4},$$

and (3.23) gives

$$\|\tan \Theta(\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1))\|_F \leq 2.3494 \cdot 10^{-4},$$

while other $\tan \Theta$ -bounds are inapplicable.

5. Conclusion. This paper presents two tan Θ -theorems, Theorem 3.5 and Theorem 3.6, for definite Hermitian matrix pairs that belong to relative perturbation theory. Theorem 3.5 gives a bound (3.23) that holds only for the Frobenius norm and requires no assumption on the spectrum. Furthermore, Theorem 3.6 gives the bound (3.24) that holds for all unitarily invariant norms, but it requires some condition on the spectrum. In several numerical examples, both bounds are tested and compared with existing tan Θ - and sin Θ -bounds. When conditions on the spectrum are satisfied, it is shown that (3.24) and (3.23) are equally sharp (see Example 4.1) and comparable to the sin Θ -bound from [6]. Otherwise, when these conditions are not satisfied (Example 4.2), the bound (3.23) is as sharp as the relative sin Θ -theorem from [6] while (3.24) cannot be applied. Additionally, in Example 4.3, it is shown that for the case where $M = I$ and $\delta M = 0$, these bounds are comparable to the tan Θ -bounds for the standard eigenvalue problem. Especially, when compared with the tan Θ -bound with relaxed conditions from [18], it can give a sharper estimate, almost equal to the exact value tan Θ . One of the reasons is that the new bounds (3.23) and (3.24) use relative gaps. Finally, we can conclude that our results are sharp enough to recognize small perturbations, and compared with existing tan Θ -bounds, they can give sharper estimates. Additionally, (3.23) can be applied in certain cases where the other tan Θ -bounds are not applicable.

Acknowledgement. This work was supported by the Croatian Science Foundation, grant no. IP-2019-04-6774, Vibration Reduction in Mechanical Systems-VIMS. The second author (Ninoslav Truhar) was supported in part by the Croatian Science Foundation under the project conduction IP-2022-10-5191.

REFERENCES

- [1] J. BARLOW AND J. DEMMEL, *Computing accurate eigensystems of scaled diagonally dominant matrices*, SIAM J. Numer. Anal., 27 (1990), pp. 762–791.
- [2] C. DAVIS AND W. M. KAHAN, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal., 7 (1970), pp. 1–46.
- [3] J. DEMMEL AND K. VESELIĆ, *Jacobi's method is more accurate than QR*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 1204–1245.
- [4] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 1996.
- [5] L. GRUBIŠIĆ, S. MIODRAGOVIĆ, AND N. TRUHAR, *Double angle theorems for definite matrix pairs*, Electron. Trans. Numer. Anal., 45 (2016), pp. 33–57.
<https://etna.ricam.oeaw.ac.at/vol.45.2016/pp33-57.dir/pp33-57.pdf>
- [6] L. GRUBIŠIĆ, N. TRUHAR, AND S. MIODRAGOVIĆ, *The rotation of eigenspaces of perturbed matrix pairs II*, Linear Multilinear Algebra, 62 (2014), pp. 1010–1031.
- [7] L. GRUBIŠIĆ, N. TRUHAR, AND K. VESELIĆ, *The rotation of eigenspaces of perturbed matrix pairs*, Linear Algebra Appl., 436 (2012), pp. 4161–4178.
- [8] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
- [9] I. C. F. IPSEN, *An overview of relative sin Θ theorems for invariant subspaces of complex matrices*, J. Comput. Appl. Math., 123 (2000), pp. 131–153.
- [10] ———, *A note on unifying absolute and relative perturbation bounds*, Linear Algebra Appl., 358 (2003), pp. 239–253.
- [11] T. KATO, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.
- [12] A. V. KNYAZEV AND M. E. ARGENTATI, *Principal angles between subspaces in an A -based scalar product: algorithms and perturbation estimates*, SIAM J. Sci. Comput., 23 (2002), pp. 2008–2040.
- [13] V. KOSTRYKIN, K. A. MAKAROV, AND A. K. MOTOVILOV, *On the existence of solutions to the operator Riccati equation and the tan Θ theorem*, Integral Equations Operator Theory, 51 (2005), pp. 121–140.
- [14] I. KUZMANOVIĆ IVIČIĆ, S. MIODRAGOVIĆ, AND M. UGRICA, *The tan θ theorem for definite matrix pairs*, Linear Multilinear Algebra, 71 (2023), pp. 2484–2498.
- [15] R.-C. LI, *On perturbations of matrix pencils with real spectra*, Math. Comp., 62 (1994), pp. 231–265.
- [16] ———, *Relative perturbation theory. II. Eigenspace and singular subspace variations*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 471–492.

- [17] P. G. MASSEY, D. STOJANOFF, AND S. ZARATE, *Majorization bounds for Ritz values of self-adjoint matrices*, SIAM J. Matrix Anal. Appl., 41 (2020), pp. 554–572.
- [18] Y. NAKATSUKASA, *The $\tan \theta$ theorem with relaxed conditions*, Linear Algebra Appl., 436 (2012), pp. 1528–1534.
- [19] G. W. STEWART AND J. G. SUN, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.
- [20] I. SLAPNIČAR AND N. TRUHAR, *Relative perturbation theory for hyperbolic eigenvalue problem*, Linear Algebra Appl., 309 (2000), pp. 57–72.
- [21] J. G. SUN, *The perturbation bounds for eigenspaces of a definite matrix-pair*, Numer. Math., 41 (1983), pp. 321–343.
- [22] N. TRUHAR AND R.-C. LI, *A $\sin 2 \theta$ theorem for graded indefinite Hermitian matrices*, Linear Algebra Appl., 359 (2003), pp. 263–276.
- [23] K. VESELIĆ, *A Jacobi eigenreduction algorithm for definite matrix pairs*, Numer. Math., 64 (1993), pp. 241–269.
- [24] ———, *Damped Oscillations of Linear Systems*, Springer, Heidelberg, 2011.
- [25] P. ZHU, M. E. ARGENTATI, AND A. V. KNYAZEV, *Bounds for the Rayleigh quotient and the spectrum of self-adjoint operators*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 244–256.
- [26] P. ZHU AND A. V. KNYAZEV, *Rayleigh-Ritz majorization error bounds of mixed type*, SIAM J. Matrix Anal. Appl., 38 (2017), pp. 30–49.