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ERROR ESTIMATES FOR GAUSS-TYPE QUADRATURE RULES FOR VARIABLE-SIGN WEIGHT FUNCTIONS*

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Dedicated to Giuseppe Rodriguez on the occasion of his 60th birthday.

Abstract. When Gauss quadrature rules are applied, the weight function is usually assumed to be nonnegative on the interval of integration. This paper considers recently introduced Gauss-type quadrature formulas with respect to weight functions that change sign in the interior of the interval of integration. To economically estimate the error of these formulas, we propose extensions based on Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss quadrature rules. Numerical examples illustrate the accuracy of the introduced error estimates.

Key words. Gauss quadrature formula, variable-sign weight function, error estimate

AMS subject classifications. 65D30, 65D32, 41A55

1. Introduction.

Denote by $[a,b]$ a finite closed real interval and by (a,b) its interior. Let $\mathcal{C}[a,b]$ represent the space of real-valued continuous functions on $[a,b]$, $\mathcal{R}[a,b]$ the space of real-valued Riemann integrable functions on $[a,b]$, and $\mathcal{P}_d$ ($d \in \mathbb{N}_0$) the subspace of real polynomials of degree at most d. A quadrature rule is said to exist if all its nodes are real.

We consider an integrand $f \in \mathcal{C}[a,b]$ and a weight function $\omega \in \mathcal{R}[a,b]$. On (a,b), we assume that ω takes the value 0 only on a set of measure zero and that it changes sign at and only at (pairwise distinct) points

$$x_k \in (a,b), \quad k = 1,2,\dots,m \quad (m \in \mathbb{N}). \tag{1.1}$$

Recently, the paper [17] proposed an n-point ($n \in \mathbb{N}$) Gauss-type quadrature rule Q_n for the approximation of the integral

$$I(f) = \int_a^b f(x)\,\omega(x)dx.$$

In this section, we briefly summarize the results from the paper [17].

Application of Q_n requires all the points (1.1) to be known, exactly or approximately. When the points (1.1) are obtained, we compute the values of the integrand f at those points $f(x_k) = y_k$, $k = 1,2,\dots,m$. Then, we choose any functions

$$\varphi_s \in \mathcal{C}[a,b], \quad s = 1,2,\dots,m, \tag{1.2}$$

such that the analytical solutions of the integrals

$$I(\varphi_s) = \int_a^b \varphi_s(x)\omega(x)dx, \quad s = 1,2,\dots,m,$$

can be (easily) found and such that for $\boldsymbol{y} = [y_1, y_2, \dots, y_m]^T$ and

$$\boldsymbol{\Phi} = \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \cdots & \varphi_m(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \cdots & \varphi_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(x_m) & \varphi_2(x_m) & \cdots & \varphi_m(x_m) \end{bmatrix}, \tag{1.3}$$

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there exists a solution $\boldsymbol{c} = [c_1, c_2, \cdots, c_m]^T$ of the $m \times m$ system of linear equations

$$\boldsymbol{\Phi c} = \boldsymbol{y}. \tag{1.4}$$

Once the functions (1.2) and one solution of the system (1.4) are determined, we define the modifier function

$$g \equiv \sum_{s=1}^{m} c_s \varphi_s. \tag{1.5}$$

It holds that $g \in \mathcal{C}[a,b]$,

$$g(x_k) = y_k, \quad k = 1, 2, \ldots, m,$$

and the integral

$$I(g) = \int_a^b g(x)\,\omega(x)dx = \sum_{s=1}^{m} c_s \int_a^b \varphi_s(x)\,\omega(x)dx \tag{1.6}$$

can be (easily) found.

Then, we introduce the modified integrand

$$\bar{f} \equiv f - g \tag{1.7}$$

and notice that it holds

$$\bar{f}(x_k) = 0, \quad k = 1, 2, \ldots, m,$$

and

$$I(f) = I(g) + I(\bar{f}).$$

This means that the given integral $I(f)$ can be represented as a sum of the integral $I(g)$ that does not cause a quadrature error and the integral $I(\bar{f})$ with the property that the points in the interior of the interval of integration at which the weight function changes sign are the zeros of its integrand.

Further, we define a real polynomial

$$q_m(x) = \pm \prod_{k=1}^{m} (x - x_k), \quad x \in \mathbb{R}, \tag{1.8}$$

and the modified weight function

$$\widetilde{\omega} \equiv q_m \omega, \tag{1.9}$$

where the plus or minus sign in (1.8) is chosen so that (1.9) is nonnegative on $[a,b]$. We consider the n-point Gauss quadrature rule with respect to the nonnegative modified weight function (1.9):

$$\begin{gathered} \widetilde{I}(\bar{f}) = \int_a^b \bar{f}(x)\,\widetilde{\omega}(x)dx = \widetilde{G}_n(\bar{f}) + \widetilde{R}_n^G(\bar{f}), \qquad \widetilde{G}_n(\bar{f}) = \sum_{i=1}^{n} \widetilde{\omega}_i^G \bar{f}(\widetilde{\tau}_i^G), \\ \widetilde{R}_n^G(p_{2n-1}) = 0, \qquad p_{2n-1} \in \mathcal{P}_{2n-1}. \end{gathered} \tag{1.10}$$

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To compute the nodes $\widetilde{\tau}_i^G$ and the weights $\widetilde{\omega}_i^G$, $i = 1, 2, \ldots, n$, of the formula (1.10), the Golub-Welsch algorithm [7] and the discrete Stieltjes procedure proposed by Gautschi [5] can be used.

If each node of the formula (1.10) differs from each point (1.1) at which the weight function ω changes sign on (a, b), i.e., if it holds

$$\widetilde{\tau}_i^G \neq x_k, \qquad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m,$$

then the integral of the modified integrand can be approximated by the quadrature rule

$$\begin{aligned} I(\bar{f}) = \int_a^b \bar{f}(x)\,\omega(x)dx = \mathcal{G}_n(\bar{f}) + R_n^{\mathcal{G}}(\bar{f}), \qquad \mathcal{G}_n(\bar{f}) = \sum_{i=1}^{n} \omega_i^{\mathcal{G}} \bar{f}(\tau_i^{\mathcal{G}}), \\ R_n^{\mathcal{G}}(q_m p_{2n-1}) = 0, \qquad p_{2n-1} \in \mathcal{P}_{2n-1}, \end{aligned} \tag{1.11}$$

where

$$\tau_i^{\mathcal{G}} = \widetilde{\tau}_i^G, \quad \omega_i^{\mathcal{G}} = \frac{\widetilde{\omega}_i^G}{q_m(\widetilde{\tau}_i^G)}, \qquad i = 1, 2, \ldots, n.$$

The nodes $\tau_i^{\mathcal{G}}$, $i = 1, 2, \ldots, n$, of the formula (1.11) are pairwise distinct and belong to the interior of the interval of integration.

To approximate the given integral, we use the quadrature rule

$$\begin{aligned} I(f) = \int_a^b f(x)\,\omega(x)dx = Q_n(f) + R_n^Q(f), \\ Q_n(f) = I(g) + \mathcal{G}_n(\bar{f}), \qquad R_n^Q(f) = R_n^{\mathcal{G}}(\bar{f}), \end{aligned} \tag{1.12}$$

where $I(g)$ is the integral of the modifier function, while $\mathcal{G}_n(\bar{f})$ and $R_n^{\mathcal{G}}(\bar{f})$ represent the quadrature sum and the remainder term, respectively, of the formula (1.11), associated with the modified integrand.

The structure of this paper is as follows. In Section 2, to estimate the error of the n-point formula (1.12), we use $(2n+1)$-point Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss quadrature rules that inherit the n nodes of formula (1.11) (all considered error estimates are obtained by following the same pattern). Section 3 is devoted to numerical examples. Some concluding remarks are given in Section 4.

2. Error estimates for Gauss-type quadrature formulas for variable-sign weight functions. Suppose that the points (1.1) at which the weight function ω changes sign in the interior of the integration interval are all determined, exactly or approximately. Also suppose that the nodes and weights of the formula (1.11) are computed. To economically estimate the error of the n-point quadrature rule (1.12), in this section we use $(2n+1)$-point Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss extensions that inherit the n nodes of the formula (1.11).

2.1. Error estimates based on the Gauss-Kronrod quadrature formula. Consider the $(2n+1)$-point Gauss-Kronrod extension of the n-point Gauss quadrature rule (1.10) for the nonnegative modified weight function (1.9), which we assume to exist:

$$\begin{aligned} \widetilde{I}(\bar{f}) = \int_a^b \bar{f}(x)\,\widetilde{\omega}(x)dx = \widetilde{K}_n(\bar{f}) + \widetilde{R}_n^K(\bar{f}), \\ \widetilde{K}_n(\bar{f}) = \sum_{i=1}^{n} \widetilde{\omega}_i^K \bar{f}(\widetilde{\tau}_i^G) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^K \bar{f}(\widetilde{\tau}_j^K), \\ \widetilde{R}_n^K(p_{3n+1}) = 0, \qquad p_{3n+1} \in \mathcal{P}_{3n+1}. \end{aligned} \tag{2.1}$$

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The formula (2.1) inherits the n nodes $\widetilde{\tau}_i^G$, $i = 1, 2, \ldots, n$, of the formula (1.10).

The nodes and weights of Gauss-Kronrod quadrature formulas can be efficiently computed by methods described in [1, 2, 9]. For more properties and references about Gauss-Kronrod quadrature rules, see Notaris [10].

If each node of the formula (2.1) differs from each point (1.1) at which the weight function ω changes sign on (a, b), i.e., if it holds

$$\begin{gathered} \widetilde{\tau}_i^G \neq x_k, \qquad \widetilde{\tau}_j^K \neq x_k, \\ i = 1, 2, \ldots, n, \quad j = n+1, n+2, \ldots, 2n+1, \quad k = 1, 2, \ldots, m, \end{gathered} \tag{2.2}$$

then we can propose a $(2n+1)$-point extension that inherit the n nodes $\tau_i^{\mathcal{G}}$, $i = 1, 2, \ldots, n$, of the formula (1.11):

$$\begin{gathered} I(\bar{f}) = \int_a^b \bar{f}(x)\,\omega(x)dx = \mathcal{K}_n(\bar{f}) + R_n^{\mathcal{K}}(\bar{f}), \\ \mathcal{K}_n(\bar{f}) = \sum_{i=1}^{n} \omega_i^{\mathcal{K}} \bar{f}(\tau_i^{\mathcal{G}}) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{K}} \bar{f}(\tau_j^{\mathcal{K}}), \\ R_n^{\mathcal{K}}(q_m p_{3n+1}) = 0, \qquad p_{3n+1} \in \mathcal{P}_{3n+1}, \end{gathered} \tag{2.3}$$

where

$$\begin{cases} \tau_i^{\mathcal{G}} = \widetilde{\tau}_i^G, & \omega_i^{\mathcal{K}} = \dfrac{\widetilde{\omega}_i^K}{q_m(\widetilde{\tau}_i^G)}, \qquad i = 1, 2, \ldots, n, \\ \tau_j^{\mathcal{K}} = \widetilde{\tau}_j^K, & \omega_j^{\mathcal{K}} = \dfrac{\widetilde{\omega}_j^K}{q_m(\widetilde{\tau}_j^K)}, \qquad j = n+1, n+2, \ldots, 2n+1. \end{cases} \tag{2.4}$$

Indeed, by (1.8) and the assumption (2.2), it holds

$$\begin{cases} q_m(\widetilde{\tau}_i^G) \neq 0, & i = 1, 2, \ldots, n, \\ q_m(\widetilde{\tau}_j^K) \neq 0, & j = n+1, n+2, \ldots, 2n+1, \end{cases} \tag{2.5}$$

from which it follows that $\omega_i^{\mathcal{K}}$, $i = 1, 2, \ldots, n$, and $\omega_j^{\mathcal{K}}$, $j = n+1, n+2, \ldots, 2n+1$, are well defined by (2.4). In view of (1.9), (2.1), (2.4), and (2.5), we obtain

$$\begin{aligned} \int_a^b q_m(x) p_{3n+1}(x)\,\omega(x)dx &= \int_a^b p_{3n+1}(x)\,\widetilde{\omega}(x)dx \\ &= \sum_{i=1}^{n} \widetilde{\omega}_i^K p_{3n+1}\left(\widetilde{\tau}_i^G\right) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^K p_{3n+1}\left(\widetilde{\tau}_j^K\right) \\ &= \sum_{i=1}^{n} \frac{\widetilde{\omega}_i^K}{q_m\left(\widetilde{\tau}_i^G\right)} q_m\left(\widetilde{\tau}_i^G\right) p_{3n+1}\left(\widetilde{\tau}_i^G\right) + \sum_{j=n+1}^{2n+1} \frac{\widetilde{\omega}_j^K}{q_m\left(\widetilde{\tau}_j^K\right)} q_m\left(\widetilde{\tau}_j^K\right) p_{3n+1}\left(\widetilde{\tau}_j^K\right) \\ &= \sum_{i=1}^{n} \omega_i^{\mathcal{K}} q_m\left(\tau_i^{\mathcal{G}}\right) p_{3n+1}\left(\tau_i^{\mathcal{G}}\right) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{K}} q_m\left(\tau_j^{\mathcal{K}}\right) p_{3n+1}\left(\tau_j^{\mathcal{K}}\right), \end{aligned}$$

which proves that the formula (2.3) is exact for all polynomials of the form $q_m p_{3n+1}$, where q_m is defined by (1.8) and $p_{3n+1} \in \mathcal{P}_{3n+1}$.

We introduce the error estimation of the formula (1.12):

$$|R_n^Q(f)| = |R_n^{\mathcal{G}}(\bar{f})| = |(I - \mathcal{G}_n)(\bar{f})| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|. \tag{2.6}$$

2.2. Error estimates based on the averaged Gauss quadrature formula. There are situations when some nodes of the Gauss-Kronrod quadrature rule are complex. This motivated the construction of alternatives to the Gauss-Kronrod formula. One of those alternatives, which always exists, is the averaged Gauss quadrature rule introduced by Laurie [8].

In this subsection, we consider the $(2n+1)$-point averaged Gauss extension that inherit the n nodes $\widetilde{\tau}_i^G$, $i=1,2,\ldots,n$, of the n-point Gauss quadrature rule (1.10):

$$\widetilde{I}(\bar{f}) = \int_a^b \bar{f}(x)\,\widetilde{\omega}(x)dx = \widetilde{L}_n(\bar{f}) + \widetilde{R}_n^L(\bar{f}),$$
$$\widetilde{L}_n(\bar{f}) = \sum_{i=1}^{n} \widetilde{\omega}_i^L \bar{f}(\widetilde{\tau}_i^G) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^L \bar{f}(\widetilde{\tau}_j^L),$$
$$\widetilde{R}_n^L(p_{2n+1}) = 0, \qquad p_{2n+1} \in \mathcal{P}_{2n+1}.$$

As in Section 2.1, if it holds

$$\widetilde{\tau}_i^G \neq x_k, \qquad \widetilde{\tau}_j^L \neq x_k,$$
$$i=1,2,\ldots,n, \quad j=n+1,n+2,\ldots,2n+1, \quad k=1,2,\ldots,m,$$

then we can propose another $(2n+1)$-point extension that inherit the n nodes $\tau_i^{\mathcal{G}}$, $i=1,2,\ldots,n$, of the formula (1.11):

$$I(\bar{f}) = \int_a^b \bar{f}(x)\,\omega(x)dx = \mathcal{L}_n(\bar{f}) + R_n^{\mathcal{L}}(\bar{f}),$$
$$\mathcal{L}_n(\bar{f}) = \sum_{i=1}^{n} \omega_i^{\mathcal{L}} \bar{f}\left(\tau_i^{\mathcal{G}}\right) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{L}} \bar{f}\left(\tau_j^{\mathcal{L}}\right),$$
$$R_n^{\mathcal{L}}\left(q_m p_{2n+1}\right) = 0, \qquad p_{2n+1} \in \mathcal{P}_{2n+1},$$

where

$$\begin{cases} \tau_i^{\mathcal{G}} = \widetilde{\tau}_i^G, & \omega_i^{\mathcal{L}} = \dfrac{\widetilde{\omega}_i^L}{q_m\left(\widetilde{\tau}_i^G\right)}, \qquad i=1,2,\ldots,n, \\ \tau_j^{\mathcal{L}} = \widetilde{\tau}_j^L, & \omega_j^{\mathcal{L}} = \dfrac{\widetilde{\omega}_j^L}{q_m\left(\widetilde{\tau}_j^L\right)}, \qquad j=n+1,n+2,\ldots,2n+1. \end{cases}$$

The error of the formula (1.12) can be also estimated by:

$$|R_n^Q(f)| = |R_n^{\mathcal{G}}(\bar{f})| = |(I-\mathcal{G}_n)(\bar{f})| \approx |(\mathcal{L}_n-\mathcal{G}_n)(\bar{f})|. \tag{2.7}$$

2.3. Error estimates based on the generalized averaged Gauss quadrature formula. Another alternative to the Gauss-Kronrod formula is the generalized averaged Gauss quadrature rule introduced by Spalević [15]; see also [14, 16]. The generalized averaged Gauss formula represents a modification of the averaged Gauss formula, it always exists, and its construction is based on certain results on positive quadrature rules by Peherstorfer [11].

Some new results on averaged and generalized averaged Gauss quadrature rules can be found in [12, 13]. For some recent results on the application of averaged and generalized averaged Gauss quadrature rules to the numerical solution of integral equations, see [3, 4].

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Consider the $(2n+1)$-point generalized averaged Gauss extension that inherit the n nodes $\widetilde{\tau}_i^G$, $i=1,2,\ldots,n$, of the n-point Gauss quadrature rule (1.10):

$$\widetilde{I}(\bar{f}) = \int_a^b \bar{f}(x)\,\widetilde{\omega}(x)dx = \widetilde{S}_n(\bar{f}) + \widetilde{R}_n^S(\bar{f}),$$

$$\widetilde{S}_n(\bar{f}) = \sum_{i=1}^{n} \widetilde{\omega}_i^S \bar{f}\left(\widetilde{\tau}_i^G\right) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^S \bar{f}\left(\widetilde{\tau}_j^S\right),$$

$$\widetilde{R}_n^S(p_{2n+2}) = 0, \qquad p_{2n+2} \in \mathcal{P}_{2n+2}.$$

Again, as in Section 2.1, if it holds

$$\widetilde{\tau}_i^G \neq x_k, \qquad \widetilde{\tau}_j^S \neq x_k,$$
$$i=1,2,\ldots,n, \quad j=n+1,n+2,\ldots,2n+1, \quad k=1,2,\ldots,m,$$

then we propose another $(2n+1)$-point extension that inherits the n nodes $\tau_i^{\mathcal{G}}$, $i=1,2,\ldots,n$, of the formula (1.11):

$$I(\bar{f}) = \int_a^b \bar{f}(x)\,\omega(x)dx = \mathcal{S}_n(\bar{f}) + R_n^{\mathcal{S}}(\bar{f}),$$

$$\mathcal{S}_n(\bar{f}) = \sum_{i=1}^{n} \omega_i^{\mathcal{S}} \bar{f}\left(\tau_i^{\mathcal{G}}\right) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{S}} \bar{f}\left(\tau_j^{\mathcal{S}}\right),$$

$$R_n^{\mathcal{S}}\left(q_m p_{2n+2}\right) = 0, \qquad p_{2n+2} \in \mathcal{P}_{2n+2},$$

where

$$\begin{cases} \tau_i^{\mathcal{G}} = \widetilde{\tau}_i^G, & \omega_i^{\mathcal{S}} = \dfrac{\widetilde{\omega}_i^S}{q_m\left(\widetilde{\tau}_i^G\right)}, \qquad i=1,2,\ldots,n, \\ \tau_j^{\mathcal{S}} = \widetilde{\tau}_j^S, & \omega_j^{\mathcal{S}} = \dfrac{\widetilde{\omega}_j^S}{q_m\left(\widetilde{\tau}_j^S\right)}, \qquad j=n+1,n+2,\ldots,2n+1. \end{cases}$$

To estimate the error of the formula (1.12), we also can use:

$$|R_n^Q(f)| = |R_n^{\mathcal{G}}(\bar{f})| = |(I-\mathcal{G}_n)(\bar{f})| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|. \tag{2.8}$$

3. Numerical tests. The present section describes numerical tests, which confirm the applicability of the error estimates (2.6), (2.7), and (2.8). The OPQ suite [6] and some codes written in MATLAB by the author of this paper are used.

EXAMPLE 3.1. Consider a simple analytically solvable integral

$$I(f) = \int_0^1 \cos^2(3\pi x)\,\sin(3\pi x)dx = \frac{2}{9\pi} \approx 0.070735530263065,$$

where $f(x) = \cos^2(3\pi x)$ is the integrand and $\omega(x) = \sin(3\pi x)$ is the weight function. In the interior of the integration interval $(0,1)$ there are exactly two points at which the weight function ω changes sign. They are

$$x_1 = \frac{1}{3}, \qquad x_2 = \frac{2}{3}.$$

It follows that the polynomial (1.8) takes the form

$$q_2(x) = (x-\tfrac{1}{3})(x-\tfrac{2}{3}),$$

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and, hence, the modified weight function (1.9) is

$$\widetilde{\omega}(x) = (x - \tfrac{1}{3})(x - \tfrac{2}{3}) \sin(3\pi x).$$

We have

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

First, for the functions (1.2), we propose

$$\varphi_1(x) = \varphi_{V,1}(x) = 1, \quad \varphi_2(x) = \varphi_{V,2}(x) = x,$$

and compute the associated integrals:

$$I(\varphi_{V,1}) = \int_0^1 \sin(3\pi x) dx = \frac{2}{3\pi},$$
$$I(\varphi_{V,2}) = \int_0^1 x \sin(3\pi x) dx = \frac{1}{3\pi}.$$

In this case, the matrix (1.3) is the Vandermonde matrix

$$\boldsymbol{\Phi} = \boldsymbol{V} = \begin{bmatrix} 1 & 1/3 \\ 1 & 2/3 \end{bmatrix},$$

while the solution of the system (1.4) is

$$\boldsymbol{c} = \boldsymbol{c_V} = \begin{bmatrix} c_{V,1} \\ c_{V,2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In view of (1.5), the modifier function is

$$g(x) = g_V(x) = c_{V,1} + c_{V,2} x = 1,$$

while for the integral (1.6) of the modifier function we obtain

$$I(g_V) = c_{V,1} I(\varphi_{V,1}) + c_{V,2} I(\varphi_{V,2}) = I(\varphi_{V,1}) \approx 0.212206590789194.$$

By (1.7), the modified integrand is

$$\bar{f}(x) = \bar{f}_V(x) = f(x) - g_V(x) = \cos^2(3\pi x) - 1,$$

and it holds

$$I(f) = I(g_V) + I(\bar{f}_V).$$

Based on (1.12), we can approximate the given integral by

$$I(f) \approx Q_n(f) = Q_{V,n}(f) = I(g_V) + \mathcal{G}_n(\bar{f}_V). \tag{3.1}$$

To estimate the error of the approximation (3.1), we use (2.6), (2.7), and (2.8), i.e.,

$$|R_n^{Q_V}(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_V)|, \tag{3.2}$$
$$|R_n^{Q_V}(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_V)|, \tag{3.3}$$
$$|R_n^{Q_V}(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_V)|. \tag{3.4}$$

TABLE 3.1
Example 3.1: Error $|R_n^{Q_V}(f)| = |(I - Q_{V,n})(f)|$ *and error estimations* $|R_n^{Q_V}(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_V)|$, $|R_n^{Q_V}(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_V)|$, *and* $|R_n^{Q_V}(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_V)|$, *for* $n = 3, 4, 5, 6, 7, 8$.

n	$\lvert(I - Q_{V,n})(f)\rvert$	$\lvert(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_V)\rvert$	$\lvert(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_V)\rvert$	$\lvert(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_V)\rvert$
3	2.550e-01	2.538e-01	2.667e-01	2.512e-01
4	1.204e-01	1.205e-01	1.066e-01	1.198e-01
5	1.404e-02	1.404e-02	1.581e-02	1.418e-02
6	3.175e-03	3.175e-03	2.979e-03	3.168e-03
7	2.568e-04	2.568e-04	2.607e-04	2.571e-04
8	2.037e-05	—	2.067e-05	2.037e-05

Table 3.1 reports the exact error $|R_n^{Q_V}(f)| = |(I - Q_{V,n})(f)|$ and the error estimates (3.2), (3.3), and (3.4), for $n = 3, 4, 5, 6, 7, 8$. For $n = 8$, it turns out that the Gauss-Kronrod quadrature rule (2.1) does not exist. It seems that all three error estimates give results similar to the exact error.

Now, for the functions (1.2), let us propose

$$\varphi_1(x) = \varphi_{\Phi,1}(x) = e^x, \quad \varphi_2(x) = \varphi_{\Phi,2}(x) = e^{2x}.$$

The associated integrals are:

$$I(\varphi_{\Phi,1}) = \int_0^1 e^x \sin(3\pi x)dx = \frac{3\pi(e+1)}{9\pi^2+1},$$
$$I(\varphi_{\Phi,2}) = \int_0^1 e^{2x} \sin(3\pi x)dx = \frac{3\pi(e^2+1)}{9\pi^2+4}.$$

In this case, the matrix (1.3) is

$$\mathbf{\Phi} = \begin{bmatrix} e^{1/3} & e^{2/3} \\ e^{2/3} & e^{4/3} \end{bmatrix} \approx \begin{bmatrix} 1.395612425086090 & 1.947734041054676 \\ 1.947734041054676 & 3.793667894683177 \end{bmatrix}.$$

As a solution of the system (1.4), we obtain

$$\boldsymbol{c} = \boldsymbol{c_\Phi} = \begin{bmatrix} c_{\Phi,1} \\ c_{\Phi,2} \end{bmatrix} \approx \begin{bmatrix} 1.229948429606381 \\ -0.367879441171442 \end{bmatrix}.$$

By (1.5), the modifier function is

$$g(x) = g_\Phi(x) = c_{\Phi,1}e^x + c_{\Phi,2}e^{2x},$$

and we can compute the integral (1.6) of the modifier function:

$$I(g_\Phi) = c_{\Phi,1}I(\varphi_{\Phi,1}) + c_{\Phi,2}I(\varphi_{\Phi,2}) \approx 0.166498072020147.$$

In view of (1.7), the modified integrand is

$$\bar{f}(x) = \bar{f}_\Phi(x) = f(x) - g_\Phi(x) = \cos^2(3\pi x) - (c_{\Phi,1}e^x + c_{\Phi,2}e^{2x}),$$

and it holds that

$$I(f) = I(g_\Phi) + I(\bar{f}_\Phi).$$

TABLE 3.2
Example 3.1: Error $|R_n^{Q_\Phi}(f)| = |(I - Q_{\Phi,n})(f)|$ *and error estimations* $|R_n^{Q_\Phi}(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_\Phi)|$, $|R_n^{Q_\Phi}(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_\Phi)|$, *and* $|R_n^{Q_\Phi}(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_\Phi)|$, *for* $n = 3, 4, 5, 6, 7, 8$.

n	$\|(I - Q_{\Phi,n})(f)\|$	$\|(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_\Phi)\|$	$\|(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_\Phi)\|$	$\|(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_\Phi)\|$
3	2.550e-01	2.538e-01	2.667e-01	2.512e-01
4	1.204e-01	1.205e-01	1.066e-01	1.198e-01
5	1.404e-02	1.404e-02	1.581e-02	1.418e-02
6	3.175e-03	3.175e-03	2.979e-03	3.168e-03
7	2.568e-04	2.568e-04	2.607e-04	2.571e-04
8	2.037e-05	—	2.067e-05	2.037e-05

Based on (1.12), the given integral also can be approximated by

$$I(f) \approx Q_n(f) = Q_{\Phi,n}(f) = I(g_\Phi) + \mathcal{G}_n(\bar{f}_\Phi). \tag{3.5}$$

Again, we use (2.6), (2.7), and (2.8) to estimate the error of the approximation (3.5), i.e.,

$$|R_n^{Q_\Phi}(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f}_\Phi)|, \tag{3.6}$$

$$|R_n^{Q_\Phi}(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f}_\Phi)|, \tag{3.7}$$

$$|R_n^{Q_\Phi}(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f}_\Phi)|. \tag{3.8}$$

The exact error $|R_n^{Q_\Phi}(f)| = |(I - Q_{\Phi,n})(f)|$ and the error estimations (3.6), (3.7), and (3.8), for $n = 3, 4, 5, 6, 7, 8$, are shown in Table 3.2 (for $n = 8$, the Gauss-Kronrod quadrature rule (2.1) does not exist). Notice that the obtained results presented in Table 3.1 and Table 3.2 are practically the same.

EXAMPLE 3.2. In this example, we consider the integral

$$I(f) = \int_{-1}^{1} e^{x^2}(e^x - 2\cos x)dx,$$

where the integrand is $f(x) = e^{x^2}$ and the weight function is $\omega(x) = e^x - 2\cos x$. There is exactly one point in the interior of the interval of integration $(-1, 1)$ at which the weight function ω changes sign, and the approximate value of that point with precision $0.5 \cdot 10^{-15}$ is

$$x_1 \approx 0.539785160809281.$$

The polynomial (1.8) and the modified weight function (1.9) are, respectively,

$$q_1(x) = x - x_1,$$
$$\widetilde{\omega}(x) = (x - x_1)(e^x - 2\cos x).$$

It holds

$$\boldsymbol{y} = y_1 = f(x_1) \approx 1.338256998448261.$$

In (1.2) we have to choose only one function—let it be

$$\varphi_1(x) = 1.$$

The analytical solution of the associated integral is

$$I(\varphi_1) = \int_{-1}^{1}(e^x - 2\cos x)dx = e - \tfrac{1}{e} - 4\sin 1 \approx -1.015481551943983.$$

TABLE 3.3
Example 3.2: Error estimations $|R_n^Q(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|$, $|R_n^Q(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})|$, *and* $|R_n^Q(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|$, *for* $n = 3, 4, 5, 6, 7, 8$.

n	$\|(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})\|$	$\|(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})\|$	$\|(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})\|$
3	8.475e-05	7.852e-05	7.649e-05
4	1.121e-06	1.718e-06	1.841e-06
5	3.328e-07	3.665e-07	3.536e-07
6	3.388e-09	3.403e-09	3.399e-09
7	—	1.267e-10	1.298e-10
8	4.241e-12	4.580e-12	4.496e-12

The matrix (1.3) is

$$\mathbf{\Phi} = 1,$$

and for the solution of the system (1.4), we obtain

$$\boldsymbol{c} = c_1 = y_1 \approx 1.338256998448261.$$

From (1.5) it follows that the modifier function is

$$g(x) = c_1,$$

while the integral (1.6) of the modifier function is

$$I(g) = c_1 I(\varphi_1) \approx -1.358975293684137.$$

From (1.7) it follows that the modified integrand is

$$\bar{f}(x) = f(x) - g(x) = e^{x^2} - c_1,$$

and it holds

$$I(f) = I(g) + I(\bar{f}).$$

In view of (1.12), the considered integral can be approximated by

$$I(f) \approx Q_n(f) = I(g) + \mathcal{G}_n(\bar{f}). \tag{3.9}$$

Similarly to the previous example, to estimate the error of approximation (3.9), we use (2.6), (2.7), and (2.8). The error estimations $|R_n^Q(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|$, $|R_n^Q(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})|$, and $|R_n^Q(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|$, for $n = 3, 4, 5, 6, 7, 8$ are shown in Table 3.3. For $n = 7$, the Gauss-Kronrod quadrature rule (2.1) does not exist. We notice that all three error estimates give similar results.

EXAMPLE 3.3. Let us now consider the integral

$$I(f) = \int_0^1 \frac{\sqrt{\cos^3 x + x}}{x^2 + 0.1 + \sqrt{e^x}} \, (\ln(x + 0.6) - \sin(11x)) \, dx,$$

where

$$f(x) = \frac{\sqrt{\cos^3 x + x}}{x^2 + 0.1 + \sqrt{e^x}}$$

is the integrand and $\omega(x) = \ln(x + 0.6) - \sin(11x)$ is the weight function. This integral was also considered in numerical experiments in [17]. There are exactly three points in the interior of the integration interval $(0, 1)$ at which the weight function ω changes sign, and the approximate values of those points with precision $0.5 \cdot 10^{-15}$ are

$$x_1 \approx 0.295639485449891, \quad x_2 \approx 0.586848729278417, \quad x_3 \approx 0.823957465020420.$$

The polynomial (1.8) is

$$q_3(x) = (x - x_1)(x - x_2)(x - x_3),$$

while the modified weight function (1.9) is

$$\widetilde{\omega}(x) = (x - x_1)(x - x_2)(x - x_3)\left(\ln(x + 0.6) - \sin(11x)\right).$$

We compute

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \approx \begin{bmatrix} 0.803554196799477 \\ 0.604337959998847 \\ 0.465989151379413 \end{bmatrix}.$$

In (1.2) let us choose

$$\varphi_1(x) = 1, \qquad \varphi_2(x) = x, \qquad \varphi_3(x) = x^2.$$

The analytical solutions of the associated integrals are

$$\begin{aligned} I(\varphi_1) &= \int_0^1 (\ln(x + 0.6) - \sin(11x))dx \\ &= \frac{88\ln\left(\frac{8}{5}\right) - 33\ln\left(\frac{3}{5}\right) + 5\cos(11) - 60}{55}, \\ &\approx -0.032005573675588, \\ I(\varphi_2) &= \int_0^1 x\,(\ln(x + 0.6) - \sin(11x))dx \\ &= \frac{6050\ln\left(\frac{8}{5}\right) + 2178\ln\left(\frac{3}{8}\right) - 100\sin(11) + 1100\cos(11) + 605}{12100}, \\ &\approx 0.117119267133809, \\ I(\varphi_3) &= \int_0^1 x^2\,(\ln(x + 0.6) - \sin(11x))dx \\ &= \frac{998250\ln\left(\frac{8}{5}\right) - 215622\ln\left(\frac{3}{8}\right) - 49500\sin(11) + 267750\cos(11) - 388145}{2994750} \\ &\approx 0.114603550863348. \end{aligned}$$

The matrix (1.3) is the Vandermonde matrix

$$\boldsymbol{\Phi} = \boldsymbol{V} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0.295639485449891 & 0.087402705357076 \\ 1 & 0.586848729278417 & 0.344391431055693 \\ 1 & 0.823957465020420 & 0.678905904162876 \end{bmatrix}.$$

TABLE 3.4
Example 3.3: Error estimations $|R_n^Q(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|$, $|R_n^Q(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})|$, *and* $|R_n^Q(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|$, *for* $n = 3, 4, 5, 6, 7, 8$.

n	$\lvert(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})\rvert$	$\lvert(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})\rvert$	$\lvert(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})\rvert$
3	—	8.120e-07	8.163e-07
4	2.921e-08	2.852e-08	2.916e-08
5	—	1.762e-09	1.756e-09
6	2.616e-11	2.849e-11	2.620e-11
7	—	2.409e-12	2.355e-12
8	—	9.609e-14	9.569e-14

For the solution of the system (1.4), we obtain

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \approx \begin{bmatrix} 1.038843170436327 \\ -0.852168640576758 \\ 0.190448621833740 \end{bmatrix}.$$

From (1.5), it follows that the modifier function is

$$g(x) = c_1 + c_2 x + c_3 x^2,$$

and the integral (1.6) of the modifier function is

$$I(g) = \sum_{s=1}^{3} c_s I(\varphi_s) \approx -0.111228049968368.$$

By (1.7), the modified integrand is

$$\bar{f}(x) = f(x) - g(x) = \frac{\sqrt{\cos^3 x + x}}{x^2 + 0.1 + \sqrt{e^x}} - (c_1 + c_2 x + c_3 x^2).$$

It holds

$$I(f) = I(g) + I(\bar{f}).$$

In view of (1.12), the considered integral can be approximated by

$$I(f) \approx Q_n(f) = I(g) + \mathcal{G}_n(\bar{f}). \tag{3.10}$$

We use (2.6), (2.7), and (2.8) to estimate the error of approximation (3.10) in this example as well.

The error estimations $|R_n^Q(f)| \approx |(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|$, $|R_n^Q(f)| \approx |(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})|$, and $|R_n^Q(f)| \approx |(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|$ are shown in Table 3.4 for $n = 3, 4, 5, 6, 7, 8$. The corresponding Gauss-Kronrod extension (2.1) exists only for $n = 4$ and $n = 6$. We notice that error estimates $|(\mathcal{L}_n - \mathcal{G}_n)(\bar{f})|$ and $|(\mathcal{S}_n - \mathcal{G}_n)(\bar{f})|$ give similar results, which are also similar to the results obtained by the error estimate $|(\mathcal{K}_n - \mathcal{G}_n)(\bar{f})|$ in cases when the Gauss-Kronrod quadrature rule (2.1) exists.

4. Conclusion. In the present paper, error estimates for the Gauss-type quadrature formula with respect to a variable-sign weight function, based on the Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss quadrature rules, are proposed. The precision and applicability of these error estimates are illustrated by numerical experiments.

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