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DISTRIBUTION RESULTS FOR A SPECIAL CLASS OF MATRIX SEQUENCES: JOINING APPROXIMATION THEORY AND ASYMPTOTIC LINEAR ALGEBRA*

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Dedicated to Lothar Reichel on the occasion of his 70th birthday.

Abstract. In a recent paper, Lubinsky proved eigenvalue distribution results for a class of Hankel matrix sequences arising in several applications, ranging from Padé approximation to orthogonal polynomials and complex analysis. The results appeared in Linear Algebra and its Applications, and indeed many of the statements, whose origin belongs to the field of approximation theory and complex analysis, contain deep results in (asymptotic) linear algebra. Here we make an analysis of a part of these findings by combining his derivation with previous results in asymptotic linear algebra, showing that the use of an already available machinery shortens considerably the considered part of the derivations. Remarks and few additional results are also provided, in the spirit of bridging (numerical and asymptotic) linear algebra results and those coming from analysis and pure approximation theory.

Key words. Hankel and Toeplitz matrix, matrix sequence, eigenvalue and singular value distribution, eigenvalue and singular value clustering

AMS subject classifications. 15B05, 15A18, 47B35

1. Introduction. In a recent paper [17] Lubinsky obtained interesting eigenvalue distribution results, in the sense of Weyl (see Definition 2.2) and with respect to special weights, for a class of Hankel matrix sequences arising in several applications, ranging from Padé approximation to orthogonal polynomials and complex analysis. We remind that for Toeplitz, Hankel, Jacobi matrix sequences, and their generalizations, such as the Generalized Locally Toeplitz (GLT) class [12, 13], including virtually any matrix sequence arising from the approximation of integro-differential and fractional operators (see [8, 12, 13] and references therein), there is significant work that has produced a series of general theoretical tools [3, 9, 14, 16]. These tools have been employed extensively for the spectral analysis of various structured and unstructured matrix sequences, also in the non-Hermitian setting as in the case of the analysis in [17]. The origin of this research line is very rich, and here we refer to the following seminal publications and to the references reported therein [7, 8, 10, 20, 21, 23, 24, 25, 26]. More specifically, in the book [8], the topic is treated in depth for Toeplitz structures from an operator theory point of view, and the same is true for [7], where additional sophisticated results regarding the banded setting are reported. Hankel structures are treated in [10], while wide generalizations of the Szegö distributional theorem for Toeplitz matrix sequences can be found in [24, 25, 26]. The papers [20, 21, 23] concern the birth of the theory of GLT matrix sequences, whose notion goes beyond the Toeplitz character and allows to describe numerous "hidden structures" that arise in applications. The references reported in [11, 12, 13] are useful for understanding the use of distribution results for fast solution methods in connection with large linear systems and large eigenvalue problems arising in applied sciences and engineering. Finally, mention has to be made to [18], where eigenvalues and pseudo-eigenvalues of Toeplitz matrices are treated: a further future research line is represented by establishing connections between spectral distribution and pseudo-spectral results in the Toeplitz and GLT setting.

The results in [17] are deep and show connections among different fields. As observed by Albrecht Böttcher in his MathSciNet review, the related findings represent a "remarkable combination of linear algebra and hard mathematical analysis".

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Here, the aim is to connect a part of the findings of Lubinsky with existing tools in asymptotic linear algebra, which were derived for tackling problems concerning the zero distribution of orthogonal polynomials, and for numerical analysis studies regarding the distribution of preconditioned matrix sequences arising in the approximation of partial differential equations. In both these applications the main problem is to estimate the distribution of a matrix sequence, which is obtained as a generic perturbation of an Hermitian matrix sequence. The related results amount to stating that the perturbed and the Hermitian matrix sequences share the same distribution whenever the non-Hermitian perturbations satisfy specific asymptotic estimates in some Schatten p-norms. In the paper by Lubinsky, one of the problems successfully handled is exactly this. Here, we show how to apply the techniques in [3, 14] to the present setting. More precisely, we shorten in a substantial manner the related mathematical derivations, at least for one of the important theorems in [17], regarding classical distribution results.

The message of the current note is an invitation to overcome the "barriers" of the different fields: indeed, as it is clear from [12, 13], the considered tools have applications in numerical analysis for the fast solution of large linear systems (see [4, 5, 15] for the connection between distribution results and convergence speed of Krylov solvers and [2] for a classical convergence analysis), in engineering for the connection between the modes of the approximating matrices and those of the differential operators [11], and in approximation theory for, e.g., the distribution of the zeros of orthogonal polynomials [9, 14, 16].

The present note is organized as follows. In Section 2 we recall the specific results in [17] that we consider here, the general tools already available in the quoted literature, and we set the notation. In Section 3 we derive in a compact way Theorem 1.2, part III in [17], while in Section 4 we draw conclusions and we add few additional findings and remarks related to the main results.

2. Notations, definitions, tools. Taken a matrix $A \in \mathbb{C}^{n\times n}$, we consider the following two classes of norms

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p ,$$

with $\|y\|_p = \left(\sum_{j=1}^n y_j^p\right)^{\frac{1}{p}}$ being the l^p vector norm and $\|A\|_p$ the associated induced p-norm on $\mathbb{C}^{n\times n}$, and

$$\|A\|_{S,p} = \left(\sum_{j=1}^{n} \sigma_j(A)^p\right)^{\frac{1}{p}}$$

being the Schatten p-norm on $\mathbb{C}^{n\times n}$ (see [6] and references therein), where $\sigma_j(A)$ is the j-th singular value of A. Thus, from the singular value decomposition of A, it follows that the Schatten ∞-norm is actually the induced Euclidean norm, namely $\|A\|_2$.

DEFINITION 2.1. *Let $\{\rho_j\}_{j\geq 0}$ be an increasing sequence of positive real numbers with limit $+\infty$. Then $\{\rho_j\}_{j\geq 0}$ is called an asymptotic comparison sequence if for each $\gamma > 0$, we have*

$$\lim_{j\to+\infty} \rho_j/j^2 = 0, \qquad \limsup_{j\to+\infty} \rho_{2j}/\rho_j < +\infty, \qquad \lim_{k\to+\infty} \max_{|j|\leq\sqrt{\gamma\rho_k}} \left|1 - \frac{\rho_{k+j}}{\rho_k}\right| = 0.$$

DEFINITION 2.2. *Let $f : D \to \mathbb{C}$ be a measurable function, defined on a measurable set $D \subset \mathbb{R}^q$ with $q \geqslant 1$, $0 < m_q(D) < \infty$, where m_q is the Lebesgue measure. Let $\mathcal{C}_0(\mathbb{K})$ be the set of continuous functions with compact support over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}_0^+\}$, and let $\{A_n\}_n$ be a*

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sequence of matrices of size n with eigenvalues $\lambda_j(A_n)$, $j = 1, \dots, n$, and singular values $\sigma_j(A_n)$, $j = 1, \dots, n$.

- $\{A_n\}_n$ *is* distributed as the pair (f, D) in the sense of the eigenvalues, *in formulae*

$$\{A_n\}_n \sim_\lambda (f, D),$$

if the following limit relation holds for all $F \in \mathcal{C}_0(\mathbb{C})$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(A_n)) = \frac{1}{m_q(D)} \int_D F(f(t))\mathrm{d}t. \tag{2.1}$$

In this case, we say that f is the (spectral) symbol *of the matrix sequence $\{A_n\}_n$.*

- $\{A_n\}_n$ *is* distributed as the pair (f, D) in the sense of the singular values, *in formulae*

$$\{A_n\}_n \sim_\sigma (f, D),$$

if the following limit relation holds for all $F \in \mathcal{C}_0(\mathbb{R}_0^+)$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(A_n)) = \frac{1}{m_q(D)} \int_D F(|f(t)|)\mathrm{d}t. \tag{2.2}$$

In this case, we say that f is the (singular value) symbol *of the matrix sequence $\{A_n\}_n$.*

REMARK 2.3. When f is continuous, an informal interpretation of the limit relation (2.1) (resp. (2.2)) is that when the matrix size is sufficiently large, the eigenvalues (resp. singular values) of A_n can be approximated by a sampling of f (resp. $|f|$) on a uniform equispaced grid of the domain D.

DEFINITION 2.4. *A matrix sequence $\{X_n\}$ is weakly clustered at $s \in \mathbb{C}$ (in the eigenvalue sense) if, for any $\epsilon > 0$, the number of eigenvalues of X_n whose distance from s is larger than ϵ is $o(n)$, with n being the size of X_n. An analogous definition can be given for the property of being weakly clustered at a closed subset S of $\mathbb{C}$, with the distance from a point replaced by the usual distance from a subset of a metric space.*

PROPOSITION 2.5. *Let $\{X_n\}$ and $\{Y_n\}$ be two Hermitian sequences with X_n, Y_n of size n, let M be a closed subset of the real line, and assume $\|X_n - Y_n\|_{S,1} = o(n)$. Then $\{X_n\}$ is weakly clustered at M if and only if the same property holds for $\{Y_n\}$.*

Proof. Taking into account the definition of weak cluster in Definition 2.4, the statement is an immediate consequence of Definition 4.3 and Lemma 4.3, item 1 in [19]. □

REMARK 2.6. The relation $\{X_n\}_n \sim_\lambda (\theta, D)$, with θ being a constant function equal to $s \in \mathbb{C}$, is equivalent to $\{X_n\}$ being weakly clustered at s (taken D as any set of positive finite Lebesgue measure in $\mathbb{R}^q$ for some $q \geq 1$).

3. Main results. As in [17], we consider a smooth sequence of complex numbers $\{a_j\}_{j\geq 0}$ in the sense that there exists $\alpha > 0$ for which

$$q_j := \frac{a_{j-1}a_{j+1}}{a_j^2} = 1 - \frac{1}{\alpha \cdot j}(1 + o(1)), \qquad j \to \infty. \tag{3.1}$$

Then, we study structured matrices of the form $A_{mn} = [a_{m-j+k}]_{1\leq j,k\leq n}$, where a_j is set to 0 when $j < 0$. Let $n \in \mathbb{N}$ and $m = m(n)$ so that there exists a fixed $R > 1$, independent of the parameter n, satisfying $n/R < m < nR$. We are interested in the study of the asymptotic localization of the spectrum of $\{A_{mn}\}_n$ and in its asymptotic distribution in the sense of Definition 2.2. In his article, Lubinsky proved deep results on the eigenvalue distribution of the sequence, when suitably scaled by an asymptotic comparison sequence $\{\rho_j\}_{j\geq 0}$, as in Definition 2.1. Specifically, (III) of Theorem 1.2 in [17] is equivalent to

$$\left\{ \frac{A_{mn}}{a_m\sqrt{\rho_m}} \right\}_n \sim_\lambda (0, D), \tag{3.2}$$

with D being any domain of finite positive Lebesgue measure in $\mathbb{R}^q$ for some $q \geq 1$ as in Definition 2.2. In other words, the matrix sequence in (3.2) is zero-distributed, and hence it is a GLT sequence with zero symbol; see [12, 13]. This result may be derived using the analysis in [14] and [3]. However, part of the derivations by Lubinsky will be used in this note, namely [17, Lemma 2.1 and Lemma 4.4]. From [17, Lemma 2.1] it follows that A_{mn}/a_m is similar to some matrix E_{mn} which is still Toeplitz. In [17, Lemma 4.4] ad hoc localization results are proven for the sequence $\{E_{mn}\}_n$ using the first Gerschgorin Theorem (see [27] and references therein) and Minmax characterization of eigenvalues of Hermitian sequences; see [6]. Here is a summary of the results:

LEMMA 3.1. *Let $\{E_{mn}\}_n$ be the Toeplitz sequence with E_{mn} defined as*

$$E_{mn} = D_{mn}^{-1} A_{mn} D_{mn}/a_m,$$

D_{mn} being a diagonal matrix whose k-th diagonal entry is given by

$$[\sqrt{q_m} a_m/a_{m+1}]^k,$$

with q_m as in (3.1). Let $E_{mn} = \mathrm{Re}(E_{mn}) + i \cdot \mathrm{Im}(E_{mn})$, where $\mathrm{Re}(E_{mn}) = (E_{mn} + E_{mn}^*)/2$ and $\mathrm{Im}(E_{mn}) = (E_{mn} - E_{mn}^*)/2i$ are the real and imaginary parts of the matrix sequence, which are structurally Hermitian. Then, for $\{\rho_m\}_m$ being an asymptotic comparison sequence as in Definition 2.1, the following statements hold:

(a) the eigenvalues of $\mathrm{Re}(E_{mn})$ are essentially nonnegative in the sense that, for $j = 1, \ldots, n$,

$$\lambda_j(\mathrm{Re}(E_{mn})) \geq -o(\sqrt{\rho_m});$$

(b) the maximum absolute value of the eigenvalues of $\mathrm{Im}(E_{mn})$ is $o(\sqrt{\rho_m})$.

Therefore, we can use (a) to deduce that the real part of the sequence $\{E_{mn}\}_n$ is zero-distributed, and (b) to deduce that the imaginary part is a low-norm correction sequence.

LEMMA 3.2. *The sequence $\{\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\}_n$ is zero-distributed in the eigenvalue sense.*

Proof. We start by estimating the trace norm, namely the Schatten 1-norm, of the sequence. First of all

$$\mathrm{tr}(\mathrm{Re}(E_{mn}/\sqrt{\rho_m})) = \mathrm{Re}(\mathrm{tr}(E_{mn}/\sqrt{\rho_m})) = \mathrm{Re}(\mathrm{tr}(A_{mn}/a_m\sqrt{\rho_m})) = n/\sqrt{\rho_m},$$

where the first equality holds trivially by the linearity of the trace and the properties of the real and imaginary parts of a matrix, the second for similarity, and the last follows from the fact that, for $j = 1, \ldots, n$,

$$[A_{mn}/a_m\sqrt{\rho_m}]_{j,j} = 1/\sqrt{\rho_m}.$$

Then, we define

$$\Lambda^+ := \sum_{\{j:\lambda_j(\mathrm{Re}(E_{mn}/\sqrt{\rho_m})>0\}} \lambda_j(\mathrm{Re}(E_{mn}/\sqrt{\rho_m})),$$

$$\Lambda^- := \sum_{\{j:\lambda_j(\mathrm{Re}(E_{mn}/\sqrt{\rho_m})<0\}} |\lambda_j(\mathrm{Re}(E_{mn}/\sqrt{\rho_m}))|,$$

being the positive and the negative mass of the eigenvalues, respectively. Since the matrix $\mathrm{Re}(E_{mn}/\sqrt{\rho_m})$ is Hermitian, its singular values are the absolute values of its eigenvalues, and hence

$$\|\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\|_{S,1} = \Lambda^+ + \Lambda^- = \mathrm{tr}(\mathrm{Re}(E_{mn}/\sqrt{\rho_m})) + 2\Lambda^- = n/\sqrt{\rho_m} + 2\Lambda^-.$$

Each term in the sum defining Λ^- is $o(\sqrt{\rho_m})/\sqrt{\rho_m} = o(1)$ by (a) in Lemma 3.1, and the sum has at most n terms. Therefore,

$$\|\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\|_{S,1} = n/\sqrt{\rho_m} + 2\Lambda^- = n/\sqrt{\rho_m} + 2no(1) = n(1/\sqrt{\rho_m} + o(1)) = o(n),$$

where we used the fact that ρ_m goes to ∞ with m and, consequently, with n. For the conclusion of the proof, we will use results from [14], that is, Proposition 2.5. More precisely, by combining the last derivations with Proposition 2.5, we infer

$$\begin{aligned} \{X_n\}_n &= \{\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\}_n, \\ \{Y_n\}_n &= \{0\}_n, \\ \|X_n - Y_n\|_{S,1} &= \|\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\|_{S,1} = o(n). \end{aligned}$$

Since $\{0\}_n$ is trivially zero-distributed, it is weakly clustered at 0. Hence, by Proposition 2.5, the same result holds for $\{\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\}_n$, and thus, by Remark 2.6, also $\{\mathrm{Re}(E_{mn}/\sqrt{\rho_m})\}_n$ is zero-distributed. ∎

LEMMA 3.3. *$E_{mn}/\sqrt{\rho_m}$ is a zero-distributed matrix sequence (in the eigenvalue sense) and therefore, by similarity, also (3.2) holds.*

In order to prove the lemma, we recall the key result in [3].

THEOREM 3.4. *Let $\{X_n\}_n$ be an Hermitian matrix sequence such that $\{X_n\}_n \sim_\lambda (f, D)$, where f is a measurable function defined on a subset of $\mathbb{R}^q$, for some $q \geq 1$, with finite positive Lebesgue measure. If $\{Y_n\}_n$ is a matrix sequence such that $\|Y_n\|_{S,2} = o(\sqrt{n})$, then $\{X_n + Y_n\}_n \sim_\lambda (f, D)$.*

Proof of Lemma 3.3. By (b) of Lemma 3.1, we know that the maximum absolute value of the eigenvalues of $\mathrm{Im}(E_{mn}/\sqrt{\rho_m})$ is $o(\sqrt{\rho_m})/\sqrt{\rho_m} = o(1)$. Here we used again the fact that ρ_m goes to ∞ with m and, as a consequence, with n. Therefore, since $\{\mathrm{Im}(E_{mn}/\sqrt{\rho_m})\}_n$ is a Hermitian matrix sequence, again the singular values are the absolute values of eigenvalues, so that

$$\left\|\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\|_2 = \max_{j=1,\dots,n} \sigma_j\left(\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right) = o(1).$$

Thus, we deduce

$$\left\|i\,\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\|_{S,2} \leq \sqrt{n}|i|\left\|\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\|_{S,\infty} = \sqrt{n}\left\|\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\|_2 = o(\sqrt{n}),$$

and finally we can use Theorem 3.4 to end the proof with

$$\{X_n\}_n = \left\{ \mathrm{Re}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\}_n,$$
$$\{Y_n\}_n = \left\{ i\,\mathrm{Im}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\}_n,$$
$$\{X_n + Y_n\}_n = \left\{ \frac{E_{mn}}{\sqrt{\rho_m}}\right\}_n.$$

Since $\left\{ \mathrm{Re}\left(\frac{E_{mn}}{\sqrt{\rho_m}}\right)\right\}_n$ is zero distributed, also $\left\{ \frac{E_{mn}}{\sqrt{\rho_m}}\right\}_n$ is zero-distributed. ∎

4. Conclusion. In a recent paper Lubinsky proved eigenvalue distribution results for a class of Hankel matrix sequences arising in several applications ranging from Padé approximation to orthogonal polynomials and complex analysis. The results appeared in Linear Algebra and its Applications and contained deep results connecting pure analysis and asymptotic linear algebra. Here we considered one of the key findings by showing that the use of the appropriate machinery shortens the derivations considerably .

Below we report some additional remarks, in the spirit of bridging (numerical and asymptotical) linear algebra results and those coming from pure approximation theory.

- In [17], while proving Theorem 3.1, the author evaluates the determinant of the matrix $B_{k-1} \in \mathbb{C}^{(k-1)\times(k-1)}$, defined as follows

$$B_{k-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix},$$

  by rearranging the rows to take it to upper triangular form. However this derivation is much more straightforward by noticing that B_{k-1} can be written as a multiple of the identity of order $k-1$ plus a rank one correction, whose eigenvalues are well-known. In particular:

$$B_{k-1} = \frac{1}{2}(I_{k-1} + e^T e),$$

  where $e \in \mathbb{C}^{k-1}$ is the row vector of all ones. The eigenvalues of $I_{k-1} + e^T e$ are 1, with multiplicity $k-2$, and $1 + ee^T = 1 + k - 1 = k$ with multiplicity 1. Therefore,

$$\det(B_{k-1}) = \det\left(\frac{1}{2}\left(I_{k-1} + e^T e\right)\right) = \frac{1}{2^{k-1}}\det\left(I_{k-1} + e^T e\right) = \frac{k}{2^{k-1}}.$$

- In relation to the Hankel matrix sequence $\{H_n(f)\}_n$ generated by a function $f \in L^1[(-\pi,\pi)^d]$, it is known that $\{H_n(f)\}_n$ is weakly clustered at zero in the singular value sense (it is indeed strongly clustered if $d = 1$ and f is continuous and 2π periodic); see [10]. The eigenvalue distribution can be derived by using majorization results (see again the beautiful book by Bhatia [6]), by employing the techniques developed in [1, 22]. Indeed the class of standard probability measures given in Definition 2.2, namely

$$\mu_n = \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn})} \delta_\lambda,$$

$\Lambda(A_{mn})$ being the spectrum of A_{mn} and δ_λ the Dirac measure centered at λ, has an intrinsic impact in describing the global behavior of the spectra of the underlying matrix sequence; see Remark 2.3 and Remark 2.4. This can be used to obtain convergence estimates for (preconditioned) Krylov methods [4, 15] when applied to the solution of large linear systems. In the case when a given matrix sequence $\{X_n\}_n$ is clustered at zero in the eigenvalue sense, i.e., $\{X_n\}_n \sim_\lambda (0, D)$, the additional new measures considered by Lubinsky may lead to additional characterizations of the clustering. This issue is worth to be investigated in future researches.

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