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GAUSS-TYPE QUADRATURE RULES WITH RESPECT TO EXTERNAL ZEROS OF THE INTEGRAND*

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Dedicated to Lothar Reichel on the occasion of his 70th birthday.

Abstract. In the present paper, we propose a Gauss-type quadrature rule into which the external zeros of the integrand (the zeros of the integrand outside the integration interval) are incorporated. This new formula with n nodes, denoted by $\mathcal{G}_n$, proves to be exact for certain polynomials of degree greater than $2n-1$ (while the Gauss quadrature formula with the same number of nodes is exact for all polynomials of degree less than or equal to $2n-1$). It turns out that $\mathcal{G}_n$ has several good properties: all its nodes are pairwise distinct and belong to the interior of the integration interval, all its weights are positive, it converges, and it is applicable both when the external zeros of the integrand are known exactly and when they are known approximately. In order to economically estimate the error of $\mathcal{G}_n$, we construct its extensions that inherit the n nodes of $\mathcal{G}_n$ and that are analogous to the Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss quadrature rules. Further, we show that $\mathcal{G}_n$ with respect to the pairwise distinct external zeros of the integrand represents a special case of the (slightly modified) Gauss quadrature formula with preassigned nodes. The accuracy of $\mathcal{G}_n$ and its extensions is confirmed by numerical experiments.

Key words. Gauss quadrature formula, external zeros of the integrand, modified weight function, quadrature error, convergence of a quadrature formula.

AMS subject classifications. 65D30, 65D32, 41A55

1. Introduction.

Assume that $[a,b]$ is a finite closed real interval ($a<b$) and let f be a real-valued continuous function on its domain D_f, where $[a,b]\subset D_f\subseteq\mathbb{R}$. Suppose that ω is a given real-valued weight function, which is non-negative on $[a,b]$ (allowed to take the value 0 only on a set of measure zero) and Riemann integrable on $[a,b]$. By $\mathbb{P}$ we denote the space of real polynomials, by $\mathbb{P}_d$ ($d\in\mathbb{N}_0$) the space of real polynomials of degree less than or equal to d, while $C[a,b]$ represents the space of real-valued continuous functions on $[a,b]$.

A quadrature rule is said to exist if all its nodes are real. Under the previously made assumptions (which are sufficient but not necessary), for each $n\in\mathbb{N}$ there exists the n-point Gauss quadrature formula, which is known to be a unique optimal interpolatory quadrature rule with a polynomial degree of exactness $2n-1$:

$$I(f)=\int_a^b f(x)\,\omega(x)dx=G_n(f)+R_n^G(f):=\sum_{i=1}^n \omega_i^G f(\tau_i^G)+R_n^G(f), \tag{1.1}$$

where the remainder term R_n^G is such that $R_n^G(p_{2n-1})=0,\ \forall p_{2n-1}\in\mathbb{P}_{2n-1}$.

All nodes τ_i^G are pairwise distinct and contained in the (open) interval (a,b), while all weights ω_i^G are positive, $i=1,2,\dots,n$. Besides, the Gauss quadrature rule (1.1) converges, i.e.,

$$\lim_{n\to\infty} R_n^G(f)=0.$$

Let $u,v\in\mathbb{P}$. Monic orthogonal polynomials with respect to the inner product

$$\langle u,v\rangle_\omega=\int_a^b u(x)v(x)\,\omega(x)dx$$

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satisfy a three-term recurrence relation whose coefficients can be computed by the Stieltjes procedure; see, e.g., [4]. These coefficients determine a symmetric tridiagonal Jacobi matrix. Golub and Welsch [11] developed an efficient algorithm for computing the nodes and weights of G_n, based on the observations that the nodes τ_i^G are the eigenvalues of the ($n \times n$ leading principal submatrix of the) Jacobi matrix and that the weights ω_i^G are proportional to the square of the first components of the corresponding normalized eigenvectors, $i = 1, 2, \ldots, n$. More details about orthogonal polynomials and Gauss quadrature rules can be found in [4, 8, 10, 12, 23].

In order to economically estimate the error of the Gauss quadrature rule (1.1), we can use $(2n+1)$-point extensions of G_n, that inherit the n nodes of G_n. These extensions (assumed to exist) are quadrature formulas of the form

$$\begin{aligned} I(f) = \int_a^b f(x)\,\omega(x)dx &= H_n(f) + R_n^H(f) \\ &:= \sum_{i=1}^{n} \omega_i^H f(\tau_i^G) + \sum_{j=n+1}^{2n+1} \omega_j^H f(\tau_j^H) + R_n^H(f), \end{aligned} \tag{1.2}$$

where τ_i^G is defined in (1.1) for each $i = 1, 2, \ldots, n$, $R_n^H(p_{d^H}) = 0$, $\forall p_{d^H} \in \mathbb{P}_{d^H}$, and $d^H > 2n-1$ is an integer, the value of which depends on the choice of extension. The formula (1.2) is commonly used to estimate the error of formula (1.1) by

$$|R_n^G(f)| = |(I - G_n)(f)| \approx |(H_n - G_n)(f)|. \tag{1.3}$$

If (1.2) represents the Gauss-Kronrod quadrature formula, we set

$$\begin{aligned} &H_n = K_n, \quad R_n^H = R_n^K, \quad d^H = d^K, \quad \omega_i^H = \omega_i^K, \quad i = 1, 2, \ldots, n, \\ &\tau_j^H = \tau_j^K, \quad \omega_j^H = \omega_j^K, \quad j = n+1, n+2, \ldots, 2n+1. \end{aligned}$$

It holds $d^K = 3n+1$. The nodes and weights of K_n can be efficiently computed by methods described in [1, 2, 14]. A nice recent discussion on many properties of Gauss-Kronrod quadrature rules is provided by Notaris [15].

There are several known situations when the nodes τ_j^K, $j = n+1, n+2, \ldots, 2n+1$, are complex. For that reason, alternatives to K_n have been developed. The first alternative to K_n is the averaged Gauss quadrature formula introduced by Laurie [13]. If (1.2) is the averaged Gauss formula, we use the notation

$$\begin{aligned} &H_n = L_n, \quad R_n^H = R_n^L, \quad d^H = d^L, \quad \omega_i^H = \omega_i^L, \quad i = 1, 2, \ldots, n, \\ &\tau_j^H = \tau_j^L, \quad \omega_j^H = \omega_j^L, \quad j = n+1, n+2, \ldots, 2n+1. \end{aligned}$$

It is valid $d^L = 2n+1$.

Spalević [21] (see also [20, 22]), following some results on the characterization of positive quadrature rules by Peherstorfer [16], proposed the generalized averaged Gauss quadrature formula, that represents a modification of L_n. If (1.2) is the generalized averaged Gauss formula, we use the notation

$$\begin{aligned} &H_n = S_n, \quad R_n^H = R_n^S, \quad d^H = d^S, \quad \omega_i^H = \omega_i^S, \quad i = 1, 2, \ldots, n, \\ &\tau_j^H = \tau_j^S, \quad \omega_j^H = \omega_j^S, \quad j = n+1, n+2, \ldots, 2n+1. \end{aligned}$$

Formula S_n is optimal among all averaged Gauss quadrature rules and it is valid $d^S = 2n+2$. In particular, if ω is an even weight function on a symmetric integration interval, then it holds $d^S = 2n+3$.

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Notice that L_n and S_n have lower polynomial degree of exactness than K_n but L_n and S_n always exist with pairwise distinct real nodes τ_j^L and τ_j^S, $j = n+1, n+2, \ldots, 2n+1$, respectively, that interlace with the Gauss nodes τ_i^G, $i = 1, 2, \ldots, n$. Moreover, all weights of L_n and S_n are positive. Besides, L_n and S_n are easier to compute than K_n (when the latter exists). Recently, Reichel and Spalević [17] gave a new representation of S_n which is analogous to the representation of L_n and whose computational complexity is equivalent to that of L_n. For some properties and applications of averaged and generalized averaged Gauss quadrature rules see Reichel and Spalević [18].

A real zero of an integrand is said to be internal if it belongs to the (closed) integration interval $[a, b]$, while a real zero of an integrand not belonging to $[a, b]$ is said to be external. In the present paper, we construct a quadrature formula and its extensions into which the external zeros of the integrand are incorporated. The approach we use is similar to that given in papers [5, 6, 9, 19], where the rational Gauss quadrature formula and its extensions are constructed.

This paper is organized as follows. In Section 2, we introduce a Gauss-type quadrature rule into which the external zeros of the integrand are incorporated, denoted by $\mathcal{G}_n$; properties and exactness of $\mathcal{G}_n$ are analyzed and its convergence is proved. The improvement achieved by $\mathcal{G}_n$ compared to the Gauss quadrature formula is further discussed as well as the difficulties encountered at the attempt to incorporate the internal zeros of the integrand into the rule $\mathcal{G}_n$. In Section 3, we analyze the remainder term of $\mathcal{G}_n$. Section 4 is devoted to the extensions of $\mathcal{G}_n$, that inherit the n nodes of $\mathcal{G}_n$. These extensions are analogous to the Gauss-Kronrod, averaged Gauss, and generalized averaged Gauss quadrature rules and can be used to economically estimate the error of $\mathcal{G}_n$. Situations when the external zeros of the integrand are known approximately are analyzed in Section 5. In Section 6, we show that $\mathcal{G}_n$ with respect to the pairwise distinct external zeros of the integrand represents a special case of the (slightly modified) Gauss quadrature rule with preassigned nodes. Numerical examples that illustrate the accuracy of $\mathcal{G}_n$ and its extensions are given in Section 7. The advantages and disadvantages of $\mathcal{G}_n$ are briefly discussed in Section 8.

2. Gauss-type quadrature formula with respect to the external zeros of the integrand.

The main purpose of this section is to construct an n-point quadrature formula

$$I(f) = \int_a^b f(x)\,\omega(x)dx = \mathcal{G}_n(f) + R_n^{\mathcal{G}}(f), \quad \mathcal{G}_n(f) = \sum_{i=1}^n \omega_i^{\mathcal{G}} f(\tau_i^{\mathcal{G}}), \tag{2.1}$$

into which the external zeros of the integrand are incorporated. The formula we obtain turns out to be exact for certain polynomials of degree higher than $2n-1$.

2.1. Incorporating the external zeros.

Let $x_1, x_2, \ldots, x_m$ be m external zeros of the integrand f (which do not have to be pairwise distinct), i.e.,

$$f(x_k) = 0, \quad x_k \in \mathbb{R} \setminus [a, b], \quad k = 1, 2, \ldots, m.$$

The integrand f is allowed to have (internal or external) zeros other than x_k, $k = 1, 2, \ldots, m$.

Notice that the product $\prod_{k=1}^m (x - x_k)$ is either positive or negative on $[a, b]$. Let q_m be a real polynomial of exact degree m having the same zeros x_k, $k = 1, 2, \ldots, m$, as the integrand f, i.e.,

$$q_m(x) = \pm \prod_{k=1}^m (x - x_k), \tag{2.2}$$

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where the plus or minus sign is chosen so that it holds

$$q_m(x) > 0 \quad \text{for} \quad x \in [a,b]. \tag{2.3}$$

It follows that

$$\widetilde{\omega} \equiv q_m \omega \tag{2.4}$$

is a real-valued non-negative function on $[a,b]$ (which can take the value 0 only on a set of measure zero). As a product of two Riemann integrable functions on $[a,b]$, $\widetilde{\omega}$ is also a Riemann integrable function on $[a,b]$. Hence, for each $n \in \mathbb{N}$ there exists the n-point Gauss quadrature formula for the modified weight function $\widetilde{\omega}$:

$$\widetilde{I}(f) = \int_a^b f(x)\,\widetilde{\omega}(x)dx = \widetilde{G}_n(f) + \widetilde{R}_n^G(f) := \sum_{i=1}^n \widetilde{\omega}_i^G f(\widetilde{\tau}_i^G) + \widetilde{R}_n^G(f), \tag{2.5}$$

with $\widetilde{R}_n^G(p_{2n-1}) = 0$, $\forall p_{2n-1} \in \mathbb{P}_{2n-1}$.

Note that we are not interested in calculating the integral $\widetilde{I}(f)$ but the nodes and weights of $\widetilde{G}_n$ will be important in the computation of the nodes and weights of $\mathcal{G}_n$. Determining the coefficients of the three-term recurrence relation with respect to the inner product

$$\langle u, v\rangle_{\widetilde{\omega}} = \int_a^b u(x)v(x)\,\widetilde{\omega}(x)dx, \quad u, v \in \mathbb{P}, \tag{2.6}$$

by the Stieltjes procedure can be difficult because of the (non-classical) weight function $\widetilde{\omega}$. In [3], Gautschi proposed the discrete Stieltjes procedure, which consists of applying the Stieltjes procedure to a discrete inner product

$$\langle u, v\rangle_N = \sum_{i=1}^N \omega_{i,N} u(\tau_{i,N})v(\tau_{i,N}), \quad N > n. \tag{2.7}$$

In view of (2.4), we obtain discretization (2.7) by

$$\tau_{i,N} = \tau_{i,N}^G, \quad \omega_{i,N} = q_m(\tau_{i,N}^G)\omega_{i,N}^G, \quad i = 1, 2, \dots, N, \tag{2.8}$$

where $\tau_{i,N}^G$ and $\omega_{i,N}^G$, $i = 1, 2, \dots, N$, are the nodes and weights of the N-point Gauss quadrature formula (1.1) with respect to the weight function ω.

Since $\widetilde{\tau}_i^G \in (a,b)$ and $x_k \in \mathbb{R} \setminus [a,b]$, it holds

$$\widetilde{\tau}_i^G \neq x_k, \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots, m,$$

and therefore

$$q_m(\widetilde{\tau}_i^G) \neq 0, \quad i = 1, 2, \dots, n. \tag{2.9}$$

From (2.3) and the assumption that $f \in C[a,b]$, it follows that $f/q_m \in C[a,b]$.

We are ready to state the following theorem.

THEOREM 2.1. *Let us define the nodes and weights of the formula* (2.1) *in terms of the nodes and weights of the formula* (2.5) *as*

$$\tau_i^{\mathcal{G}} = \widetilde{\tau}_i^G, \quad \omega_i^{\mathcal{G}} = \frac{\widetilde{\omega}_i^G}{q_m(\widetilde{\tau}_i^G)}, \quad i = 1, 2, \dots, n. \tag{2.10}$$

Then, the nodes of formula (2.1) *given in* (2.10) *satisfy*

$$\tau_i^{\mathcal{G}} \in (a,b), \quad i = 1,2,\ldots,n, \quad \textit{and} \quad \tau_i^{\mathcal{G}} \neq \tau_j^{\mathcal{G}}, \quad i,j = 1,2,\ldots,n, \quad i \neq j, \tag{2.11}$$

while its weights are such that

$$\omega_i^{\mathcal{G}} > 0, \quad i = 1,2,\ldots,n. \tag{2.12}$$

The remainder term of quadrature rule (2.1) *satisfies*

$$R_n^{\mathcal{G}}(q_m p_{2n-1}) = 0, \quad p_{2n-1} \in \mathbb{P}_{2n-1}, \tag{2.13}$$

as well as

$$R_n^{\mathcal{G}}(f) = \widetilde{R}_n^G\left(f/q_m\right). \tag{2.14}$$

Moreover, quadrature rule (2.1) *converges,* i.e.,

$$\lim_{n\to\infty} R_n^{\mathcal{G}}(f) = 0. \tag{2.15}$$

Proof. First notice that $\omega_i^{\mathcal{G}}$, $i = 1,2,\ldots,n$, is well defined by (2.10), since it holds (2.9). The nodes of the Gauss quadrature formula (2.5) satisfy

$$\widetilde{\tau}_i^G \in (a,b), \quad i = 1,2,\ldots,n, \quad \text{and} \quad \widetilde{\tau}_i^G \neq \widetilde{\tau}_j^G, \quad i,j = 1,2,\ldots,n, \quad i \neq j. \tag{2.16}$$

The statement (2.11) follows from taking into account (2.10) and (2.16).

The weights of the Gauss quadrature formula (2.5) satisfy

$$\widetilde{\omega}_i^G > 0, \quad i = 1,2,\ldots,n. \tag{2.17}$$

From (2.3), (2.10), (2.16), and (2.17) follows the statement (2.12).

By (2.4), (2.5), (2.9), and (2.10), we obtain

$$\begin{aligned}\int_a^b q_m(x)p_{2n-1}(x)\,\omega(x)dx &= \int_a^b p_{2n-1}(x)\,\widetilde{\omega}(x)dx = \sum_{i=1}^n \widetilde{\omega}_i^G p_{2n-1}(\widetilde{\tau}_i^G)\\ &= \sum_{i=1}^n \frac{\widetilde{\omega}_i^G}{q_m(\widetilde{\tau}_i^G)} q_m(\widetilde{\tau}_i^G)p_{2n-1}(\widetilde{\tau}_i^G) = \sum_{i=1}^n \omega_i^{\mathcal{G}} q_m(\tau_i^{\mathcal{G}})p_{2n-1}(\tau_i^{\mathcal{G}}),\end{aligned}$$

which proves the statement (2.13). In view of (2.9) and (2.10), it holds

$$\mathcal{G}_n(f) = \sum_{i=1}^n \omega_i^{\mathcal{G}} f(\tau_i^{\mathcal{G}}) = \sum_{i=1}^n \omega_i^{\mathcal{G}} q_m(\widetilde{\tau}_i^G)\frac{f(\tau_i^{\mathcal{G}})}{q_m(\widetilde{\tau}_i^G)} = \sum_{i=1}^n \widetilde{\omega}_i^G \frac{f(\widetilde{\tau}_i^G)}{q_m(\widetilde{\tau}_i^G)}. \tag{2.18}$$

Considering (2.3), which implies $q_m(x) \neq 0$ for $x \in [a,b]$, it also holds

$$I(f) = \int_a^b f(x)\,\omega(x)dx = \int_a^b \frac{f(x)}{q_m(x)} q_m(x)\,\omega(x)dx = \int_a^b \frac{f(x)}{q_m(x)}\,\widetilde{\omega}(x)dx. \tag{2.19}$$

Since by (2.1)

$$R_n^{\mathcal{G}}(f) = I(f) - \mathcal{G}_n(f),$$

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and since by (2.5)

$$\widetilde{R}_n^G(f/q_m) = \int_a^b \frac{f(x)}{q_m(x)}\,\widetilde{\omega}(x)dx - \sum_{i=1}^{n} \widetilde{\omega}_i^G \frac{f(\widetilde{\tau}_i^G)}{q_m(\widetilde{\tau}_i^G)},$$

from (2.18) and (2.19) follows the statement (2.14). Since $f/q_m \in C[a,b]$, the Gauss quadrature formula (2.5) for the integrand f/q_m converges, i.e.,

$$\lim_{n\to\infty} \widetilde{R}_n^G(f/q_m) = 0,$$

and the statement (2.15) follows from the previously proved statement (2.14). □

Let us explain the improvement achieved by the formula (2.1), (2.10). The Gauss quadrature rule (1.1) with n nodes is exact for all polynomials of $\mathbb{P}_{2n-1}$. In situations when some properties of the integrand (such as its external zeros) are known, it may not be necessary for an n-point quadrature formula to be exact on the whole space $\mathbb{P}_{2n-1}$ but only on its subset which contains only polynomials similar to the integrand, i.e., polynomials with the same properties as the integrand. For an integer $d \geq m$, let

$$\mathbb{Q}_d = \{q_m p_{d-m} :\ p_{d-m} \in \mathbb{P}_{d-m}\}$$

be the subset of $\mathbb{P}_d$ containing only polynomials with the same (external) zeros x_k, $k = 1,2,\ldots,m$, as the integrand f. The degree of polynomials in $\mathbb{Q}_{2n-1}$ (excluding the zero polynomial) is greater than or equal to m and less than or equal to $2n-1$ and the Gauss rule (1.1) with n nodes is exact on $\mathbb{Q}_{2n-1}$. For $s > 2n-1$ there is no guarantee that the Gauss rule (1.1) with n nodes is exact on $\mathbb{Q}_s$, since $\mathbb{Q}_s \not\subseteq \mathbb{P}_{2n-1}$. On the other hand, from Theorem 2.1, it follows that formula (2.1), (2.10) with also n nodes is exact on $\mathbb{Q}_{2n-1+m}$ ($\supset \mathbb{Q}_{2n-1}$). The degrees of polynomials in $\mathbb{Q}_{2n-1+m}$ (excluding the zero polynomial) are greater than or equal to m and less than or equal to $2n-1+m$ ($> 2n-1$). To achieve a polynomial degree of exactness $2n-1+m$, the Gauss rule (1.1) requires $n + \lfloor \frac{m+1}{2} \rfloor$ nodes ($\lfloor\cdot\rfloor$ denotes the integer part of a number).

2.2. Incorporating the internal zeros. Let $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{\overline{m}}$ be $\overline{m}$ internal zeros of the integrand f (which do not have to be pairwise distinct), i.e.,

$$f(\overline{x}_l) = 0, \quad \overline{x}_l \in [a,b], \quad l = 1,2,\ldots,\overline{m}.$$

If we wanted to incorporate the internal zeros of the integrand into the quadrature formula (2.1), (2.10), we would encounter several difficulties. The zeros $\overline{x}_l$, $l = 1,2,\ldots,\overline{m}$, which differ from a and b, would have to be of even multiplicities because otherwise the polynomial

$$\overline{q}_m(x) = \pm \prod_{l=1}^{\overline{m}} (x - \overline{x}_l),$$

as well as the modified weight function

$$\overline{\omega} \equiv \overline{q}_m \omega,$$

changes sign in the interior of $[a,b]$. Moreover, the division by zero in the expression analogous to (2.10) (and in the expression similar to (2.18)) would be possible, since there is no guarantee that each node $\widetilde{\tau}_i^G$ of the Gauss quadrature formula (2.5) differs from each internal zero $\overline{x}_l$ of the integrand f, i.e.,

$$\widetilde{\tau}_i^G \neq \overline{x}_l, \quad i = 1,2,\ldots,n, \quad l = 1,2,\ldots,\overline{m},$$

is not guaranteed. Besides, the division by $\overline{q}_m$ in the expressions analogous to (2.19), and thus the convergence of quadrature formula (2.1), (2.10), would become questionable. Also notice that $f/\overline{q}_m \notin C[a,b]$.

3. The remainder term. In this section, the remainder term $R_n^{\mathcal{G}}$ of the quadrature formula (2.1), (2.10) is analyzed. By $\|\cdot\|$, we denote the maximum norm on $[a, b]$, i.e.,

$$\|f\| = \|f\|_{\infty,[a,b]} = \max_{x\in[a,b]} |f(x)|, \quad f \in C[a, b]. \tag{3.1}$$

Let $p \in \mathbb{P}_{2n-1}$ be an arbitrary polynomial. Suppose that there exists $\widehat{p} \in \mathbb{P}_{2n-1}$ such that $q_m\widehat{p}$ is the best approximation of f by polynomials of the form $q_m p$ in the maximum norm. By

$$\epsilon(f) = \min_{p\in\mathbb{P}_{2n-1}} \|f - q_m p\| = \|f - q_m \widehat{p}\| \tag{3.2}$$

we denote the error of best approximation of f by polynomials of the form $q_m p$ in the maximum norm.

From (2.1) and (2.13), it follows

$$\int_a^b q_m(x)p(x)\,\omega(x)dx = \sum_{i=1}^{n} \omega_i^{\mathcal{G}} q_m(\tau_i^{\mathcal{G}})p(\tau_i^{\mathcal{G}}). \tag{3.3}$$

Notice that by (2.12) it holds $|\omega_i^{\mathcal{G}}| = \omega_i^{\mathcal{G}}$, $i = 1, 2, \dots, n$.

In view of (2.1) and (3.3), we have

$$\begin{aligned}\left|R_n^{\mathcal{G}}(f)\right| &= \Bigg|\int_a^b f(x)\,\omega(x)dx - \int_a^b q_m(x)p(x)\,\omega(x)dx \\ &\qquad + \sum_{i=1}^{n} \omega_i^{\mathcal{G}} q_m(\tau_i^{\mathcal{G}})p(\tau_i^{\mathcal{G}}) - \sum_{i=1}^{n} \omega_i^{\mathcal{G}} f(\tau_i^{\mathcal{G}})\Bigg| \\ &\le \int_a^b |f(x) - q_m(x)p(x)|\,\omega(x)dx + \sum_{i=1}^{n} \omega_i^{\mathcal{G}} \left|q_m(\tau_i^{\mathcal{G}})p(\tau_i^{\mathcal{G}}) - f(\tau_i^{\mathcal{G}})\right|. \end{aligned}\tag{3.4}$$

Since (3.4) holds for every $p \in \mathbb{P}_{2n-1}$, it also holds for $p \equiv \widehat{p}$ and we obtain

$$\begin{aligned}\left|R_n^{\mathcal{G}}(f)\right| &\le \int_a^b |f(x) - q_m(x)\widehat{p}(x)|\,\omega(x)dx + \sum_{i=1}^{n} \omega_i^{\mathcal{G}} \left|q_m(\tau_i^{\mathcal{G}})\widehat{p}(\tau_i^{\mathcal{G}}) - f(\tau_i^{\mathcal{G}})\right| \\ &\le \|f - q_m\widehat{p}\| \int_a^b \omega(x)dx + \|f - q_m\widehat{p}\| \sum_{i=1}^{n} \omega_i^{\mathcal{G}}, \end{aligned}$$

from which, by (3.2), it follows

$$\left|R_n^{\mathcal{G}}(f)\right| \le \epsilon(f) \left(\int_a^b \omega(x)dx + \sum_{i=1}^{n} \omega_i^{\mathcal{G}}\right). \tag{3.5}$$

Result (3.5) suggests that if $\epsilon(f)$ is small, i.e., if f can be well approximated by polynomials of the form $q_m p$, then the quadrature error $R_n^{\mathcal{G}}(f)$ is also small. Since f and q_m have the same m zeros x_k, $k = 1, 2, \dots, m$, the approximation of f by polynomials of the form $q_m p$ might be better than the approximation of f by polynomials from $\mathbb{P}_{2n-1}$ – this is observed in several (but not in all) numerical examples in Section 7.

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4. Extensions and error estimates. In order for economically estimate the error of formula (2.1), (2.10), in this section, we introduce its $(2n+1)$-point extensions that inherit the n nodes from $\mathcal{G}_n$. These extensions are analogous to formula (1.2).

With an aim to construct a $(2n+1)$-point quadrature formula of the form

$$\begin{aligned} I(f) = \int_a^b f(x)\,\omega(x)dx = \mathcal{H}_n(f) + R_n^{\mathcal{H}}(f), \\ := \sum_{i=1}^{n} \omega_i^{\mathcal{H}} f(\tau_i^{\mathcal{G}}) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{H}} f(\tau_j^{\mathcal{H}}) + R_n^{\mathcal{H}}(f), \end{aligned} \tag{4.1}$$

where $\tau_i^{\mathcal{G}}$, $i = 1, 2, \ldots, n$, are defined in (2.10), we consider a $(2n+1)$-point extension (assumed to exist) of the Gauss quadrature formula (2.5) for the modified weight function $\widetilde{\omega}$ (given by (2.4)), i.e.,

$$\begin{aligned} \widetilde{I}(f) = \int_a^b f(x)\,\widetilde{\omega}(x)dx = \widetilde{H}_n(f) + \widetilde{R}_n^{H}(f), \\ := \sum_{i=1}^{n} \widetilde{\omega}_i^{H} f(\widetilde{\tau}_i^{G}) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^{H} f(\widetilde{\tau}_j^{H}) + \widetilde{R}_n^{H}(f), \end{aligned} \tag{4.2}$$

where $\widetilde{\tau}_i^{G}$, $i = 1, 2, \ldots, n$, is defined in (2.5), $\widetilde{R}_n^{H}(p_{\widetilde{d}^H}) = 0$, $\forall p_{\widetilde{d}^H} \in \mathbb{P}_{\widetilde{d}^H}$, and $\widetilde{d}^H > 2n-1$ is an integer, the value of which depends on the choice of extension.

If $\widetilde{H}_n$ represents the Gauss-Kronrod (assumed to exist), averaged Gauss, or generalized averaged Gauss quadrature rule, then the computation of its nodes and weights can also be done with respect to the discrete inner product (2.7), where discretization is obtained by (2.8).

The proof of the following theorem is analogous to the proof of statements (2.13) and (2.14) of Theorem 2.1.

THEOREM 4.1. *Assume that the quadrature rule* (4.2) *exists and that each node* $\widetilde{\tau}_j^{H}$ *differs from each external zero* x_k *of the integrand* f, i.e.,

$$\widetilde{\tau}_j^{H} \neq x_k, \quad j = n+1, n+2, \ldots, 2n+1, \quad k = 1, 2, \ldots, m. \tag{4.3}$$

Setting the nodes and weights of the formula (4.1) *in terms of the nodes and weights of the formula* (4.2) *as*

$$\begin{aligned} \tau_i^{\mathcal{G}} &= \widetilde{\tau}_i^{G}, \qquad \omega_i^{\mathcal{H}} = \frac{\widetilde{\omega}_i^{H}}{q_m(\widetilde{\tau}_i^{G})}, \quad i = 1, 2, \ldots, n, \\ \tau_j^{\mathcal{H}} &= \widetilde{\tau}_j^{H}, \qquad \omega_j^{\mathcal{H}} = \frac{\widetilde{\omega}_j^{H}}{q_m(\widetilde{\tau}_j^{H})}, \quad j = n+1, n+2, \ldots, 2n+1, \end{aligned} \tag{4.4}$$

then the remainder term of quadrature rule (4.1) *satisfies*

$$R_n^{\mathcal{H}}(q_m p_{\widetilde{d}^H}) = 0, \quad p_{\widetilde{d}^H} \in \mathbb{P}_{\widetilde{d}^H},$$

as well as

$$R_n^{\mathcal{H}}(f) = \widetilde{R}_n^{H}(f/q_m).$$

We introduce the error estimation of formula (2.1), (2.10) by formula (4.1), (4.4), which is analogous to the error estimation (1.3):

$$|R_n^{\mathcal{G}}(f)| = |(I - \mathcal{G}_n)(f)| \approx |(\mathcal{H}_n - \mathcal{G}_n)(f)|. \tag{4.5}$$

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If (4.2) is the Gauss-Kronrod quadrature formula (assumed to exist), then we denote

$$\begin{aligned}&\widetilde{H}_n = \widetilde{K}_n, \quad \widetilde{R}_n^H = \widetilde{R}_n^K, \quad \widetilde{d}^H = \widetilde{d}^K = 3n+1, \quad \widetilde{\omega}_i^H = \widetilde{\omega}_i^K, \quad i = 1, 2, \ldots, n,\\ &\widetilde{\tau}_j^H = \widetilde{\tau}_j^K, \quad \widetilde{\omega}_j^H = \widetilde{\omega}_j^K, \quad j = n+1, n+2, \ldots, 2n+1,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{H}_n = \mathcal{K}_n, \quad R_n^{\mathcal{H}} = R_n^{\mathcal{K}}, \quad \omega_i^{\mathcal{H}} = \omega_i^{\mathcal{K}}, \quad i = 1, 2, \ldots, n,\\ &\tau_j^{\mathcal{H}} = \tau_j^{\mathcal{K}}, \quad \omega_j^{\mathcal{H}} = \omega_j^{\mathcal{K}}, \quad j = n+1, n+2, \ldots, 2n+1.\end{aligned}$$

If (4.2) represents the averaged Gauss quadrature formula, then we write

$$\begin{aligned}&\widetilde{H}_n = \widetilde{L}_n, \quad \widetilde{R}_n^H = \widetilde{R}_n^L, \quad \widetilde{d}^H = \widetilde{d}^L = 2n+1, \quad \widetilde{\omega}_i^H = \widetilde{\omega}_i^L, \quad i = 1, 2, \ldots, n,\\ &\widetilde{\tau}_j^H = \widetilde{\tau}_j^L, \quad \widetilde{\omega}_j^H = \widetilde{\omega}_j^L, \quad j = n+1, n+2, \ldots, 2n+1,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{H}_n = \mathcal{L}_n, \quad R_n^{\mathcal{H}} = R_n^{\mathcal{L}}, \quad \omega_i^{\mathcal{H}} = \omega_i^{\mathcal{L}}, \quad i = 1, 2, \ldots, n,\\ &\tau_j^{\mathcal{H}} = \tau_j^{\mathcal{L}}, \quad \omega_j^{\mathcal{H}} = \omega_j^{\mathcal{L}}, \quad j = n+1, n+2, \ldots, 2n+1.\end{aligned}$$

Since $\widetilde{L}_n$ always exists, the assumption on the existence of quadrature rule (4.2) in Theorem 4.1 can be omitted in this case. The nodes $\widetilde{\tau}_j^L$, $j = n+1, n+2, \ldots, 2n+1$, interlace the nodes $\widetilde{\tau}_i^G$, $i = 1, 2, \ldots, n$. This means that $\widetilde{L}_n$ can have no more than two nodes outside $[a, b]$ (that could coincide with some of the external zeros of the integrand f and therefore with the zeros of the polynomial q_m). Without loss of generality, assume that $\widetilde{\tau}_{n+1}^L$ and $\widetilde{\tau}_{2n+1}^L$ are those two nodes. In this case, condition (4.3) comes down to

$$\widetilde{\tau}_{n+1}^L \neq x_k \quad \text{and} \quad \widetilde{\tau}_{2n+1}^L \neq x_k, \quad k = 1, 2, \ldots, m.$$

If (4.2) represents the generalized averaged Gauss quadrature formula, we write

$$\begin{aligned}&\widetilde{H}_n = \widetilde{S}_n, \quad \widetilde{R}_n^H = \widetilde{R}_n^S, \quad \widetilde{d}^H = \widetilde{d}^S = 2n+2, \quad \widetilde{\omega}_i^H = \widetilde{\omega}_i^S, \quad i = 1, 2, \ldots, n,\\ &\widetilde{\tau}_j^H = \widetilde{\tau}_j^S, \quad \widetilde{\omega}_j^H = \widetilde{\omega}_j^S, \quad j = n+1, n+2, \ldots, 2n+1,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{H}_n = \mathcal{S}_n, \quad R_n^{\mathcal{H}} = R_n^{\mathcal{S}}, \quad \omega_i^{\mathcal{H}} = \omega_i^{\mathcal{S}}, \quad i = 1, 2, \ldots, n,\\ &\tau_j^{\mathcal{H}} = \tau_j^{\mathcal{S}}, \quad \omega_j^{\mathcal{H}} = \omega_j^{\mathcal{S}}, \quad j = n+1, n+2, \ldots, 2n+1.\end{aligned}$$

The assumption on the existence of quadrature rule (4.2) in Theorem 4.1 can be omitted also in this case because $\widetilde{S}_n$ also always exists. The nodes $\widetilde{\tau}_j^S$, $j = n+1, n+2, \ldots, 2n+1$, interlace with the nodes $\widetilde{\tau}_i^G$, $i = 1, 2, \ldots, n$, and $\widetilde{S}_n$ can have no more than two nodes outside $[a, b]$. Since we can assume (without loss of generality) that $\widetilde{\tau}_{n+1}^S$ and $\widetilde{\tau}_{2n+1}^S$ are those two nodes, condition (4.3) in this case comes down to

$$\widetilde{\tau}_{n+1}^S \neq x_k \quad \text{and} \quad \widetilde{\tau}_{2n+1}^S \neq x_k, \quad k = 1, 2, \ldots, m.$$

In situations when quadrature rule $\widetilde{H}_n$ is internal (i.e. when all its nodes belong to the (closed) integration interval), then condition (4.3) in Theorem 4.1 can be omitted.

5. Situations when the external zeros of the integrand are approximated. Denote by x_k the exact values of the external zeros of the integrand f but assume that only their approximations x_k^* are known:

$$x_k \approx x_k^*, \quad x_k^* \in \mathbb{R} \setminus [a,b], \quad k = 1,2,\dots,m.$$

5.1. Gauss-type quadrature formula with respect to the approximated external zeros of the integrand. Replacing the exact values x_k with the approximated values x_k^*, $k = 1,2,\dots,m$, means that instead of the polynomial (2.2), we consider its approximation

$$q_m(x) \approx q_m^*(x) = \pm \prod_{k=1}^{m} (x - x_k^*), \tag{5.1}$$

where plus or minus sign in (5.1) is chosen so that it holds

$$q_m^*(x) > 0 \quad \text{for} \quad x \in [a,b].$$

The modified weight function (2.4) is replaced by its approximation

$$\widetilde{\omega} \approx \widetilde{\omega}^* \equiv q_m^* \omega,$$

while formula (2.5) is replaced by the n-point Gauss quadrature formula for the approximated modified weight function $\widetilde{\omega}^*$:

$$\widetilde{I}^*(f) = \int_a^b f(x)\,\widetilde{\omega}^*(x)dx = \widetilde{G}_n^*(f) + \widetilde{R}_n^{G^*}(f), \quad \widetilde{G}_n^*(f) = \sum_{i=1}^{n} \widetilde{\omega}_i^{G^*} f(\widetilde{\tau}_i^{G^*}), \tag{5.2}$$

with

$$\widetilde{R}_n^{G^*}(p_{2n-1}) = 0, \quad p_{2n-1} \in \mathbb{P}_{2n-1}.$$

Instead of the inner product (2.6), we consider the inner product

$$\langle u, v \rangle_{\widetilde{\omega}^*} = \int_a^b u(x)v(x)\,\widetilde{\omega}^*(x)dx, \quad u, v \in \mathbb{P},$$

while discretization (2.7) is obtained by

$$\tau_{i,N} = \tau_{i,N}^G, \quad \omega_{i,N} = q_m^*(\tau_{i,N}^G)\omega_{i,N}^G, \quad i = 1,2,\dots,N, \tag{5.3}$$

where $\tau_{i,N}^G$ and $\omega_{i,N}^G$, $i = 1,2,\dots,N$, are the nodes and weights of the N-point Gauss quadrature formula (1.1) with respect to the weight function ω. Instead of formula (2.1), we construct an n-point quadrature formula

$$I(f) = \int_a^b f(x)\,\omega(x)dx = \mathcal{G}_n^*(f) + R_n^{\mathcal{G}^*}(f), \quad \mathcal{G}_n^*(f) = \sum_{i=1}^{n} \omega_i^{\mathcal{G}^*} f(\tau_i^{\mathcal{G}^*}), \tag{5.4}$$

into which the approximated external zeros of the integrand are incorporated. Such formula has properties described in the following theorem, which can be proved analogously as Theorem 2.1.

THEOREM 5.1. *Setting the nodes and weights of the formula* (5.4) *in terms of the nodes and weights of the formula* (5.2) *as*

$$\tau_i^{\mathcal{G}^*} = \widetilde{\tau}_i^{G^*}, \quad \omega_i^{\mathcal{G}^*} = \frac{\widetilde{\omega}_i^{G^*}}{q_m^*(\widetilde{\tau}_i^{G^*})}, \quad i = 1,2,\dots,n, \tag{5.5}$$

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then the nodes of formula (5.4) *satisfy*

$$\tau_i^{\mathcal{G}^*} \in (a,b), \quad i = 1,2,\ldots,n, \quad \textit{and} \quad \tau_i^{\mathcal{G}^*} \neq \tau_j^{\mathcal{G}^*}, \quad i,j = 1,2,\ldots,n, \quad i \neq j,$$

while its weights are such that

$$\omega_i^{\mathcal{G}^*} > 0, \quad i = 1,2,\ldots,n.$$

The remainder term of quadrature rule (5.4) *satisfies*

$$R_n^{\mathcal{G}^*}(q_m^* p_{2n-1}) = 0, \quad p_{2n-1} \in \mathbb{P}_{2n-1},$$

as well as

$$R_n^{\mathcal{G}^*}(f) = \widetilde{R}_n^{G^*}\left(f/q_m^*\right),$$

and quadrature rule (5.4) *converges,* i.e.,

$$\lim_{n\to\infty} R_n^{\mathcal{G}^*}(f) = 0.$$

5.2. The remainder term with respect to the approximated external zeros of the integrand. Let $p \in \mathbb{P}_{2n-1}$ be an arbitrary polynomial and assume that there exists $\widehat{p}^* \in \mathbb{P}_{2n-1}$ such that $q_m^*\widehat{p}^*$ is the best approximation of f by polynomials of the form $q_m^* p$ in the maximum norm (3.1). By

$$\epsilon^*(f) = \min_{p\in\mathbb{P}_{2n-1}} \|f - q_m^* p\| = \|f - q_m^*\widehat{p}^*\|,$$

we denote the error of best approximation of f by polynomials of the form $q_m^* p$ in the maximum norm (3.1). Analogously as in Section 3, we obtain

$$\left|R_n^{\mathcal{G}^*}(f)\right| \leq \epsilon^*(f)\left(\int_a^b \omega(x)dx + \sum_{i=1}^n \omega_i^{\mathcal{G}^*}\right). \tag{5.6}$$

Result (5.6) suggests that if $\epsilon^*(f)$ is small (which means that the integrand f can be well approximated by polynomials of the form $q_m^* p$), then the remainder term $R_n^{\mathcal{G}^*}(f)$ of quadrature formula (5.4), (5.5) should also be small.

5.3. Extensions and error estimates with respect to the approximated external zeros of the integrand. In order to economically estimate the error of formula (5.4), (5.5), we introduce its $(2n+1)$-point extensions that inherit the n nodes from $\mathcal{G}_n^*$. Since the external zeros of the integrand are approximated, formula (4.2) will be replaced by the $(2n+1)$-point extension (assumed to exist) of the Gauss quadrature formula (5.2) for the approximated modified weight function $\widetilde{\omega}^*$:

$$\begin{aligned} \widetilde{I}^*(f) = \int_a^b f(x)\,\widetilde{\omega}^*(x)dx &= \widetilde{H}_n^*(f) + \widetilde{R}_n^{H^*}(f) \\ &:= \sum_{i=1}^n \widetilde{\omega}_i^{H^*} f(\widetilde{\tau}_i^{G^*}) + \sum_{j=n+1}^{2n+1} \widetilde{\omega}_j^{H^*} f(\widetilde{\tau}_j^{H^*}) + \widetilde{R}_n^{H^*}(f), \end{aligned} \tag{5.7}$$

where $\widetilde{\tau}_i^{G^*}$, $i = 1,2,\ldots,n$, is defined in (5.2), $\widetilde{R}_n^{H^*}(p_{\widetilde{d}^{H^*}}) = 0$, $\forall p_{\widetilde{d}^{H^*}} \in \mathbb{P}_{\widetilde{d}^{H^*}}$, and $\widetilde{d}^{H^*} > 2n-1$ is an integer, the value of which depends on the choice of extension. If (5.7) represents the Gauss-Kronrod (assumed to exist), averaged Gauss, or generalized averaged Gauss quadrature rule, then the computation of its nodes and weights can also be done with respect to the discrete inner product (2.7), where discretization is obtained by (5.3). Instead of formula (4.1), we construct a $(2n+1)$-point quadrature formula

$$
\begin{aligned}
I(f) = \int_a^b f(x)\,\omega(x)dx &= \mathcal{H}_n^*(f) + R_n^{\mathcal{H}^*}(f) \\
&:= \sum_{i=1}^{n} \omega_i^{\mathcal{H}^*} f(\tau_i^{\mathcal{G}^*}) + \sum_{j=n+1}^{2n+1} \omega_j^{\mathcal{H}^*} f(\tau_j^{\mathcal{H}^*}) + R_n^{\mathcal{H}^*}(f),
\end{aligned} \tag{5.8}
$$

where $\tau_i^{\mathcal{G}^*}$, $i = 1, 2, \ldots, n$, are defined in (5.5). In the following theorem, which can be proved analogously as Theorem 4.1, we describe the properties of the formula (5.8).

THEOREM 5.2. *Assume that the quadrature rule* (5.7) *exists and that each node* $\widetilde{\tau}_j^{H^*}$ *differs from each approximated external zero* x_k^* *of the integrand* f, i.e.,

$$\widetilde{\tau}_j^{H^*} \neq x_k^*, \quad j = n+1, n+2, \ldots, 2n+1, \quad k = 1, 2, \ldots, m.$$

Defining the nodes and weights of the formula (5.8) *in terms of the nodes and weights of the formula* (5.7) *as*

$$
\begin{aligned}
\tau_i^{\mathcal{G}^*} &= \widetilde{\tau}_i^{G^*}, \qquad \omega_i^{\mathcal{H}^*} = \frac{\widetilde{\omega}_i^{H^*}}{q_m^*(\widetilde{\tau}_i^{G^*})}, \quad i = 1, 2, \ldots, n, \\
\tau_j^{\mathcal{H}^*} &= \widetilde{\tau}_j^{H^*}, \qquad \omega_j^{\mathcal{H}^*} = \frac{\widetilde{\omega}_j^{H^*}}{q_m^*(\widetilde{\tau}_j^{H^*})}, \quad j = n+1, n+2, \ldots, 2n+1,
\end{aligned} \tag{5.9}
$$

then the remainder term of quadrature rule (5.8) *satisfies*

$$R_n^{\mathcal{H}^*}(q_m^* p_{\widetilde{d}^{H^*}}) = 0, \quad p_{\widetilde{d}^{H^*}} \in \mathbb{P}_{\widetilde{d}^{H^*}},$$

as well as

$$R_n^{\mathcal{H}^*}(f) = \widetilde{R}_n^{H^*}(f/q_m^*).$$

The error of formula (5.4), (5.5) can be estimated by formula (5.8), (5.9):

$$|R_n^{\mathcal{G}^*}(f)| = |(I - \mathcal{G}_n^*)(f)| \approx |(\mathcal{H}_n^* - \mathcal{G}_n^*)(f)|. \tag{5.10}$$

If (5.7) is the Gauss-Kronrod (assumed to exist), averaged Gauss, or generalized averaged Gauss quadrature rule, then we set $\mathcal{H}_n^* = \mathcal{K}_n^*$, $\mathcal{H}_n^* = \mathcal{L}_n^*$, and $\mathcal{H}_n^* = \mathcal{S}_n^*$, respectively.

6. Connection with the Christoffel quadrature formula. In the present section, we assume that the external zeros of the integrand are pairwise distinct and show that in this situation the quadrature formula (2.1), (2.10) can be obtained as a special case of the (slightly modified) Gauss quadrature rule with preassigned nodes, which is also called a Christoffel quadrature rule.

Let $\rho_k^C \in \mathbb{R} \setminus [a, b]$, $k = 1, 2, \ldots, m$, be pairwise distinct fixed (preassigned) nodes of the $(n+m)$-point Christoffel quadrature formula

$$
\begin{aligned}
I(f) = \int_a^b f(x)\,\omega(x)dx &= C_n(f) + R_n^C(f) \\
&:= \sum_{i=1}^{n} \omega_i^C f(\tau_i^C) + \sum_{k=1}^{m} \sigma_k^C f(\rho_k^C) + R_n^C(f),
\end{aligned} \tag{6.1}
$$

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with

$$R_n^C(p_{2n-1+m}) = 0, \quad p_{2n-1+m} \in \mathbb{P}_{2n-1+m}.$$

Define

$$\omega^C(x) = \prod_{i=1}^{n}(x - \tau_i^C), \quad \sigma^C(x) = \pm \prod_{k=1}^{m}(x - \rho_k^C),$$

where the plus or minus sign is chosen so that it holds $\sigma^C(x) > 0$ for $x \in [a, b]$. Free nodes τ_i^C, $i = 1, 2, \ldots, n$, are equal to the nodes of the Gauss quadrature formula for the weight function $\sigma^C\omega$ (on the integration interval $[a, b]$), while the weights can be computed as

$$\omega_i^C = \int_a^b \frac{\omega^C(x)\sigma^C(x)}{[\omega^C]'(\tau_i^C)\sigma^C(\tau_i^C)(x - \tau_i^C)}\, \omega(x)dx, \quad i = 1, 2, \ldots, n, \tag{6.2}$$

$$\sigma_k^C = \int_a^b \frac{\omega^C(x)\sigma^C(x)}{\omega^C(\rho_k^C)[\sigma^C]'(\rho_k^C)(x - \rho_k^C)}\, \omega(x)dx, \quad k = 1, 2, \ldots, m. \tag{6.3}$$

The Christoffel quadrature formula is modified here by setting $\rho_k^C \in \mathbb{R} \setminus [a, b]$ instead of $\rho_k^C \in \mathbb{R} \setminus (a, b)$, $k = 1, 2, \ldots, m$, and by setting $\sigma^C(x) > 0$ instead of $\sigma^C(x) \geq 0$ for $x \in [a, b]$; see [4, 12] for more details about the Christoffel quadrature rules.

If we choose the (pairwise distinct) external zeros of the integrand to be preassigned nodes, i.e.,

$$\rho_k^C = x_k, \quad k = 1, 2, \ldots, m, \tag{6.4}$$

then it holds

$$f(\rho_k^C) = 0, \quad k = 1, 2, \ldots, m,$$

and the second sum in (6.1) disappears, which means that the weights (6.3) do not have to be computed.

For the choice (6.4) of preassigned nodes, it holds

$$\sigma^C \equiv q_m, \tag{6.5}$$

as well as $\sigma^C\omega \equiv q_m\omega \equiv \widetilde{\omega}$. Therefore, the free nodes of Christoffel quadrature formula (6.1) are equal to the nodes of the Gauss quadrature formula (2.5) for the modified weight function $\widetilde{\omega}$, which means that they are also equal to the nodes of quadrature formula (2.1), (2.10), i.e., it holds

$$\tau_i^C = \widetilde{\tau}_i^G = \tau_i^{\mathcal{G}}, \quad i = 1, 2, \ldots, n. \tag{6.6}$$

If let $\widetilde{\omega}^G(x) = \prod_{i=1}^{n}(x - \widetilde{\tau}_i^G)$, then from (6.6) it follows

$$\omega^C \equiv \widetilde{\omega}^G.$$

The weights of the Gauss quadrature formula (2.5) satisfy

$$\widetilde{\omega}_i^G = \int_a^b \widetilde{l}_i^G(x)\, \widetilde{\omega}(x)dx, \quad i = 1, 2, \ldots, n,$$

where

$$\widetilde{l}_i^G(x) = \prod_{\substack{h=1,\\ h\neq i}}^{n} \frac{x-\widetilde{\tau}_h^G}{\widetilde{\tau}_i^G-\widetilde{\tau}_h^G} = \frac{\widetilde{\omega}^G(x)}{[\widetilde{\omega}^G]'(\widetilde{\tau}_i^G)(x-\widetilde{\tau}_i^G)}, \quad i=1,2,\ldots,n. \tag{6.7}$$

From (2.4), (2.10), (6.2), and (6.5)-(6.7), it follows for each $i=1,\ldots,n$

$$\begin{aligned}\omega_i^C &= \int_a^b \frac{\widetilde{\omega}^G(x)q_m(x)}{[\widetilde{\omega}^G]'(\widetilde{\tau}_i^G)q_m(\widetilde{\tau}_i^G)(x-\widetilde{\tau}_i^G)}\,\omega(x)dx = \frac{1}{q_m(\widetilde{\tau}_i^G)}\int_a^b \widetilde{l}_i^G(x)\,\widetilde{\omega}(x)dx\\ &= \frac{\widetilde{\omega}_i^G}{q_m(\widetilde{\tau}_i^G)} = \omega_i^{\mathcal{G}}.\end{aligned}$$

The endpoints a and b of the integration interval $[a,b]$ are not allowed to be the preassigned nodes because in that case the division by q_m in (2.19), and thus the convergence of the quadrature formula (2.1), (2.10), would become questionable.

To summarize,

$$\tau_i^C = \tau_i^{\mathcal{G}}, \quad \omega_i^C = \omega_i^{\mathcal{G}}, \quad i=1,2,\ldots,n.$$

On the other hand, notice that

$$R_n^C(p_{2n-1+m}) = 0 \quad \text{for} \quad p_{2n-1+m}\in\mathbb{P}_{2n-1+m},$$

while

$$R_n^{\mathcal{G}}(q_m p_{2n-1}) = 0 \quad \text{for} \quad p_{2n-1}\in\mathbb{P}_{2n-1},$$

i.e., it appears that the space on which $\mathcal{G}_n$ is exact is a subset of the space on which C_n is it. The reason is the fact that C_n is the $(n+m)$-point formula whose construction does not depend on an integrand. However, if m preassigned nodes coincide with the external zeros of the integrand, then C_n can be considered as the n-point formula which is the same as $\mathcal{G}_n$ (the construction of $\mathcal{G}_n$ does depend on the integrand). In the n-point formula C_n, the external zeros of the integrand are incorporated as the preassigned nodes, while in the n-point formula $\mathcal{G}_n$, the external zeros of the integrand are incorporated as the property of the integrand.

Let us notice that when the external zeros of the integrand are approximated, then the Gauss-type quadrature rule $\mathcal{G}_n^*$ clearly differs from the Chistoffel quadrature rule C_n.

7. Numerical tests. The present section is devoted to numerical experiments. We used the OPQ suite [7] and some codes written in MATLAB by the author of this paper. The computation is done with 16 significant decimal digits. For comparison, at the beginning of each example, the value of the considered integral is displayed with 15 significant digits after the decimal point.

EXAMPLE 7.1. In this example, we illustrate the accuracy of quadrature rules (2.1), (2.10) and (5.4), (5.5) and compare with the accuracy of the Gauss quadrature rule (1.1).

Let $\omega\equiv 1$ and consider a simple analytically solvable integral

$$I = \int_0^1 \cos^2 x\, dx = \frac{2+\sin 2}{4} \approx 0.727324356706420.$$

All zeros of the integrand $\cos^2 x$ are external, double, and take the form

$$(2t+1)\pi/2, \quad t\in\mathbb{Z}. \tag{7.1}$$

TABLE 7.1
Example 7.1: Errors $|I - G_n|$ and $|I - \mathcal{G}_n|$ (the external zeros of the integrand are known exactly).

$$I = \int_0^1 \cos^2 x \, dx \approx 0.727324356706420$$

n	$\lvert I - G_n\rvert$	$\lvert I - \mathcal{G}_n\rvert$ $q_1 = \frac{\pi}{2} - x$	$\lvert I - \mathcal{G}_n\rvert$ $q_1 = \frac{\pi}{2} + x$	$\lvert I - \mathcal{G}_n\rvert$ $q_2 = (\frac{\pi}{2} - x)(\frac{\pi}{2} + x)$	$\lvert I - \mathcal{G}_n\rvert$ $q_2 = (\frac{\pi}{2} - x)^2$
3	8.318e–06	4.090e–06	4.032e–06	1.451e–06	1.647e–07
4	3.795e–08	1.438e–08	1.709e–08	4.097e–09	7.018e–10
5	1.069e–10	3.286e–11	4.364e–11	7.795e–12	1.660e–12

TABLE 7.2
Example 7.1: Error $|I - \mathcal{G}_4^*|$ (the external zeros of the integrand are known approximately).

$$I = \int_0^1 \cos^2 x \, dx \approx 0.727324356706420, \quad |I - G_4| = 3.795\text{e–}08$$

δ	x_1^*	x_2^*	$\lvert I - \mathcal{G}_4^*\rvert$
0	$\pi/2$	$-\pi/2$	4.097e–09
$0.5 \cdot 10^{-7}$	1.5707963	-1.5707963	4.097e–09
$0.5 \cdot 10^{-5}$	1.57080	-1.57080	4.097e–09
$0.5 \cdot 10^{-3}$	1.571	-1.571	4.099e–09
$0.5 \cdot 10^{-2}$	1.57	-1.57	4.100e–09
$0.5 \cdot 10^{-1}$	1.6	-1.6	1.280e–08

In Table 7.1 the error $|I - G_n|$ of the Gauss quadrature formula (1.1), and the error $|I - \mathcal{G}_n|$ of quadrature formula (2.1), (2.10) are shown for different choices of incorporated zeros (7.1) and for $n = 3, 4, 5$. The choices of incorporated zeros and corresponding polynomials q_m are:

$$\begin{aligned}
&x_1 = \pi/2, && q_1 = q_1(x) = \pi/2 - x;\\
&x_1 = -\pi/2, && q_1 = q_1(x) = \pi/2 + x;\\
&x_1 = \pi/2, \quad x_2 = -\pi/2, && q_2 = q_2(x) = (\pi/2 - x)(\pi/2 + x);\\
&x_1 = x_2 = \pi/2, && q_2 = q_2(x) = (\pi/2 - x)^2.
\end{aligned}$$

Notice that in all cases, formulas $\mathcal{G}_n$ turn out to be more accurate than formula G_n. Besides, formulas $\mathcal{G}_n$ into which two zeros are incorporated show better accuracy than formulas $\mathcal{G}_n$ into which only one zero is incorporated.

Consider now the situation that corresponds to the choice of incorporated zeros $x_1 = \pi/2$ and $x_2 = -\pi/2$ but assume that $x_1 \approx x_1^*$ and $x_2 \approx x_2^*$ are given approximately and suppose there is given $\delta \geq 0$ such that it holds

$$|x_k - x_k^*| \leq \delta, \quad k = 1, 2.$$

In Table 7.2 the errors $|I - \mathcal{G}_4^*|$ are shown, first for $\delta = 0$ (which means that we know the exact values x_1 and x_2 of incorporated zeros), and then for different choices of $\delta > 0$. Notice that the obtained results for $\delta > 0$ are close to the result obtained for $\delta = 0$, except when $\delta = 0.5 \cdot 10^{-1}$ but even in that case the error $|I - \mathcal{G}_4^*|$ is less than the error $|I - G_4|$.

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TABLE 7.3
Example 7.2: Errors $|I-\mathcal{G}_n|$, $|I-\mathcal{K}_n|$, $|I-\mathcal{L}_n|$, and $|I-\mathcal{S}_n|$, as well as errors $|I-G_n|$, $|I-K_n|$, $|I-L_n|$, and $|I-S_n|$.

$$I=\int_0^1\left(x+\frac{1}{10}\right)^{55/2}dx\approx 0.530697042044031$$

n	$\lvert I-\mathcal{G}_n\rvert$	$\lvert I-\mathcal{K}_n\rvert$	$\lvert I-\mathcal{L}_n\rvert$	$\lvert I-\mathcal{S}_n\rvert$
3	2.544e–01	5.290e–04	2.680e–03	1.139e–03
4	8.589e–02	1.913e–05	2.131e–04	2.656e–05
5	1.925e–02	6.101e–08	1.490e–05	1.715e–08
n	$\lvert I-G_n\rvert$	$\lvert I-K_n\rvert$	$\lvert I-L_n\rvert$	$\lvert I-S_n\rvert$
3	3.352e–01	1.659e–03	5.959e–03	3.716e–03
4	1.324e–01	5.861e–05	4.446e–04	2.084e–04
5	3.428e–02	2.248e–07	3.042e–05	9.619e–06

EXAMPLE 7.2. The purpose of the present example is to illustrate the precision of error estimate (4.5) and to compare the accuracy of the quadrature rule (4.1), (4.4) with the accuracy of the corresponding quadrature rule (1.2).

As in the previous example, let $\omega\equiv 1$. We consider a simple analytically solvable integral

$$I=\int_0^1\left(x+\frac{1}{10}\right)^{55/2}dx=\frac{2}{57}\left(\left(\frac{11}{10}\right)^{57/2}-\left(\frac{1}{10}\right)^{57/2}\right)\approx 0.530697042044031.$$

Notice that the integrand $(x+1/10)^{55/2}$ has a unique (external) zero $x_1=-1/10$. We incorporate this zero into quadrature rules $\mathcal{G}_n$, $\mathcal{K}_n$, $\mathcal{L}_n$, and $\mathcal{S}_n$. In Table 7.3 the errors

$$|I-\mathcal{G}_n|,\quad |I-\mathcal{K}_n|,\quad |I-\mathcal{L}_n|,\quad \text{and}\quad |I-\mathcal{S}_n|,$$

as well as (for comparing) the errors

$$|I-G_n|,\quad |I-K_n|,\quad |I-L_n|,\quad \text{and}\quad |I-S_n|,$$

for $n=3,4,5$, are shown, while in Table 7.4 the error estimations

$$|\mathcal{K}_n-\mathcal{G}_n|,\quad |\mathcal{L}_n-\mathcal{G}_n|,\quad \text{and}\quad |\mathcal{S}_n-\mathcal{G}_n|,$$

as well as the estimations

$$|K_n-G_n|,\quad |L_n-G_n|,\quad \text{and}\quad |S_n-G_n|,$$

also for $n=3,4,5$ are given. It turns out that $|\mathcal{K}_n-\mathcal{G}_n|$, $|\mathcal{L}_n-\mathcal{G}_n|$, and $|\mathcal{S}_n-\mathcal{G}_n|$ give good error estimates of $|I-\mathcal{G}_n|$. If we compare the errors $|I-G_n|$, $|I-K_n|$, $|I-L_n|$, and $|I-S_n|$ with $|I-\mathcal{G}_n|$, $|I-\mathcal{K}_n|$, $|I-\mathcal{L}_n|$, and $|I-\mathcal{S}_n|$, respectively, we notice that incorporating the external zero $x_1=-1/10$ improves in all considered cases.

EXAMPLE 7.3. In this example, we consider the integrand whose zeros cannot be determined analytically but must be computed by some numerical method. Then we illustrate the accuracy of quadrature rule (5.4), (5.5) using error estimate (5.10) and compare it to the accuracy of quadrature rule (1.1) obtained using error estimate (1.3).

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TABLE 7.4
Example 7.2: Error estimations $|\mathcal{K}_n-\mathcal{G}_n|$, $|\mathcal{L}_n-\mathcal{G}_n|$, and $|\mathcal{S}_n-\mathcal{G}_n|$, as well as error estimations $|K_n-G_n|$, $|L_n-G_n|$, and $|S_n-G_n|$.

$$I=\int_0^1\left(x+\frac{1}{10}\right)^{55/2}dx\approx 0.530697042044031$$

n	$\lvert\mathcal{K}_n-\mathcal{G}_n\rvert$	$\lvert\mathcal{L}_n-\mathcal{G}_n\rvert$	$\lvert\mathcal{S}_n-\mathcal{G}_n\rvert$
3	2.549e–01	2.571e–01	2.556e–01
4	8.591e–02	8.610e–02	8.592e–02
5	1.925e–02	1.926e–02	1.925e–02
n	$\lvert K_n-G_n\rvert$	$\lvert L_n-G_n\rvert$	$\lvert S_n-G_n\rvert$
3	3.369e–01	3.412e–01	3.390e–01
4	1.324e–01	1.328e–01	1.326e–01
5	3.428e–02	3.431e–02	3.429e–02

Let $f(x)=6\sin x-x^3-0.2$ be the integrand, $\omega(x)=\sqrt{1-x}$ the weight function, and $[-1,1]$ the integration interval. We consider the integral

$$I=\int_{-1}^1(6\sin x-x^3-0.2)\sqrt{1-x}dx\approx -2.181300514422565.$$

The integrand f has two external zeros,

$$x_1\approx x_1^*=-1.81878\quad\text{and}\quad x_2\approx x_2^*=1.78273,$$

where

$$|x_k-x_k^*|\le 0.5\cdot 10^{-5},\quad k=1,2.$$

The errors

$$|I-\mathcal{G}_2^*|,\quad |I-\mathcal{K}_2^*|,\quad |I-\mathcal{L}_2^*|,\quad\text{and}\quad |I-\mathcal{S}_2^*|,$$

as well as (for comparison) the errors

$$|I-G_2|,\quad |I-K_2|,\quad |I-L_2|,\quad\text{and}\quad |I-S_2|,$$

are shown in Table 7.5, while the error estimations

$$|\mathcal{K}_2^*-\mathcal{G}_2^*|,\quad |\mathcal{L}_2^*-\mathcal{G}_2^*|,\quad\text{and}\quad |\mathcal{S}_2^*-\mathcal{G}_2^*|,$$

as well as the error estimations

$$|K_2-G_2|,\quad |L_2-G_2|,\quad\text{and}\quad |S_2-G_2|,$$

are shown in Table 7.6. For instance, notice that when only x_1^* or x_2^* is incorporated, then formula $\mathcal{G}_2^*$ is less accurate than formula G_2; on the other hand, when both x_1^* and x_2^* are incorporated, then $\mathcal{G}_2^*$ is more accurate than G_2.

EXAMPLE 7.4. The aim of this example is to compare the Christoffel quadrature formula (6.1) with the Gauss-type quadrature rule (5.4), (5.5) into which approximated external zeros of the integrand are incorporated.

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TABLE 7.5
Example 7.3: Errors $|I-\mathcal{G}_2^*|$, $|I-\mathcal{K}_2^*|$, $|I-\mathcal{L}_2^*|$, and $|I-\mathcal{S}_2^*|$, as well as errors $|I-G_2|$, $|I-K_2|$, $|I-L_2|$, and $|I-S_2|$.

$$I=\int_{-1}^{1}(6\sin x-x^3-0.2)\sqrt{1-x}dx\approx -2.181300514422565$$

Incorporated zero(s)	$\lvert I-\mathcal{G}_2^*\rvert$	$\lvert I-\mathcal{K}_2^*\rvert$	$\lvert I-\mathcal{L}_2^*\rvert$	$\lvert I-\mathcal{S}_2^*\rvert$
$x_1^*=-1.81878$	9.809e–03	2.016e–08	1.427e–06	7.011e–08
$x_2^*=1.78273$	1.007e–02	2.669e–08	2.704e–08	4.479e–08
$x_1^*=-1.81878$, $x_2^*=1.78273$	1.959e–04	8.443e–10	5.601e–08	4.649e–08
	$\lvert I-G_2\rvert$	$\lvert I-K_2\rvert$	$\lvert I-L_2\rvert$	$\lvert I-S_2\rvert$
	2.904e–03	3.665e–08	9.083e–07	6.500e–07

TABLE 7.6
Example 7.3: Error estimations $|\mathcal{K}_2^*-\mathcal{G}_2^*|$, $|\mathcal{L}_2^*-\mathcal{G}_2^*|$, and $|\mathcal{S}_2^*-\mathcal{G}_2^*|$, as well as error estimations $|K_2-G_2|$, $|L_2-G_2|$, and $|S_2-G_2|$.

$$I=\int_{-1}^{1}(6\sin x-x^3-0.2)\sqrt{1-x}dx\approx -2.181300514422565$$

Incorporated zero(s)	$\lvert\mathcal{K}_2^*-\mathcal{G}_2^*\rvert$	$\lvert\mathcal{L}_2^*-\mathcal{G}_2^*\rvert$	$\lvert\mathcal{S}_2^*-\mathcal{G}_2^*\rvert$
$x_1^*=-1.81878$	9.809e–03	9.808e–03	9.809e–03
$x_2^*=1.78273$	1.007e–02	1.007e–02	1.007e–02
$x_1^*=-1.81878$, $x_2^*=1.78273$	1.959e–04	1.959e–04	1.959e–04
	$\lvert K_2-G_2\rvert$	$\lvert L_2-G_2\rvert$	$\lvert S_2-G_2\rvert$
	2.904e–03	2.903e–03	2.903e–03

For the weight function $\omega\equiv 1$, consider a simple analytically solvable integral

$$I=\int_0^1\left(100-e^{3x}\right)dx=\frac{301-e^3}{3}\approx 93.638154358937456.$$

The integrand $100-e^{3x}$ has a unique (external) zero

$$x_1=\frac{\ln 100}{3}\approx 1.53506=x_1^*,$$

where

$$|x_1-x_1^*|\leq 0.5\cdot 10^{-5}.$$

In Table 7.7 the errors $|I-G_n|$, $|I-C_n|$, and $|I-\mathcal{G}_n^*|$ for $n=3,4,5$ are shown. We notice that C_n and $\mathcal{G}_n^*$ are more accurate than G_n but there is no significant difference in accuracy between C_n and $\mathcal{G}_n^*$.

8. Conclusions. In this paper, the quadrature rule $\mathcal{G}_n$ with respect to the external zeros of the integrand is constructed. As we have seen, all nodes of $\mathcal{G}_n$ are pairwise distinct and belong to the interior of the integration interval, all its weights are positive, it converges, it is applicable when the external zeros of the integrand are known exactly as well as when they are

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TABLE 7.7
Example 7.4: Errors $|I - G_n|$, $|I - C_n|$, *and* $|I - \mathcal{G}_n^*|$.

$$I = \int_0^1 (100 - e^{3x})\, dx \approx 93.638154358937456$$

n	$\lvert I - G_n\rvert$	$\lvert I - C_n\rvert$	$\lvert I - \mathcal{G}_n^*\rvert$
3	1.735e–03	1.063e–03	1.063e–03
4	1.748e–05	7.664e–06	7.640e–06
5	1.094e–07	3.715e–08	3.552e–08

known approximately, and with n nodes it is exact for certain polynomials of degree greater than $2n - 1$.

One flaw of $\mathcal{G}_n$ is that it depends on the integrand f, i.e., on its external zeros x_k, $k = 1, 2, \ldots, m$, and once calculated nodes and weights can be used only for other integrands which also have the external zeros x_k, $k = 1, 2, \ldots, m$. However, the idea of $\mathcal{G}_n$ was not to make the constructed formula applicable for all integrands. The idea was to use known or easily determined external zeros of a given integrand to improve the accuracy which would be obtained by the Gauss quadrature formula.

Another flaw of $\mathcal{G}_n$ is that it could have a high computational cost if we have to apply certain numerical methods to determine the external zeros of f. In such situations, it might be better to use the Gauss quadrature formula with more than n nodes to achieve the desired accuracy. However, if the external zeros of f are trivial to find, or the numerical method for determining them does not require a high computational cost (recall that we do not have to know all the external zeros of f to construct $\mathcal{G}_n$), or if we have some a prior knowledge about the external zeros of f, then $\mathcal{G}_n$ should be efficient and practical for use.

Notice that results in Section 5 suggest that the introduced formulas are not necessarily related to the external zeros of f. The main point seems to be that the polynomial $q_m^* \widehat{p}^*$ must provide a good approximation of f.

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