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A PRODUCT INTEGRATION RULE FOR HYPERSINGULAR INTEGRALS ON $(0, +\infty)$*

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Dedicated to Walter Gautschi on the occasion of his 90th birthday

Abstract. In the present paper we propose a product integration rule for hypersingular integrals on the positive semi-axis. The rule is based on an approximation of the density function f by a suitable truncated Lagrange polynomial. We discuss theoretical aspects by proving stability and convergence of the procedure for density functions f belonging to weighted uniform spaces. Moreover, we give some computational details for the effective construction of the rule coefficients. For the sake of completeness, we present some numerical tests that support the theoretical estimates and some comparisons with other numerical methods.

Key words. Hadamard finite part integrals, approximation by polynomials, orthogonal polynomials, product integration rules

AMS subject classifications. 65D32, 65R20, 41A10

1. Introduction. This paper deals with the approximation of integral transforms of the type

$$\mathcal{H}_p(fu_\gamma, t) = \ne\!\!\!\!\!\!\int_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}} u_\gamma(x)dx, \qquad t > 0,\ \ p \in \mathbb{N}, \qquad u_\gamma(x) = e^{-\frac{x}{2}} x^\gamma, \quad \gamma \geq 0, \tag{1.1}$$

where the integral on the right-hand side is defined as the finite part in the Hadamard sense. Integrals of this kind are also called "hypersingular integrals" and arise in many contexts such as singular and hypersingular boundary integral equations, which are tools for modeling many phenomena in different branches of the applied sciences (see for instance [1, 11, 19, 25] and the references therein).

Recently in [5, 6, 7] we have proposed several different methods for computing the integrals (1.1), each of them useful according to the scope and/or depending on the smoothness of the density function f. In [6] we have considered a truncated Gaussian rule suitably modified to avoid the severe numerical cancellation arising when t is "close" to a Gaussian node. Whenever the simultaneous approximation of $\mathcal{H}_0(fu_\gamma, t), \mathcal{H}_1(fu_\gamma, t), \ldots, \mathcal{H}_p(fu_\gamma, t)$ is needed, the methods proposed in [5] and [7] can be more appropriate. Among them, for smoother functions, with a more expensive procedure it is reasonable to use the method in [7], while in presence of less regular functions, a local low-degree polynomial is better suited and cheaper [5]. In any case, all of the previous methods are based on the following decomposition

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commonly adopted by many other authors:

$$\int_0^{+\infty} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x)dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \fint_0^{+\infty} \frac{u_\gamma(x)}{(x-t)^{p+1-k}} dx.$$

This approach requires the derivatives of f or, alternatively with an additional effort, a suitable approximation of them (see [7]).

Here we introduce a product integration rule on the basis of a straightforward approximation of f by a truncated Lagrange polynomial essentially based on Laguerre zeros, thus obtaining

$$\mathcal{H}_p(fu_\gamma, t) = \sum_{k=1}^{j} f(x_k)\mathcal{A}_k(t) + e_{m,p}(fu_\gamma, t) =: \mathcal{H}_{p,m}(fu_\gamma, t) + e_{m,p}(fu_\gamma, t),$$

where the index $j = j(m) \leq m$ is the truncation index, and its role will be clarified later. By this way, no derivatives of the density function f are needed. We prove stability and convergence of the formula $\mathcal{H}_{p,m}(fu_\gamma, t)$, and we provide some error estimates for functions f in weighted uniform spaces of Zygmund type. We remark that "truncated" processes with respect to Laguerre weights were introduced in numerical quadrature by Mastroianni and Monegato [13, 14] and successively applied to different kinds of integrals (for instance [2, 16]), whereas "truncated" Lagrange polynomial sequences on the semi-axis were introduced in [10, 12] (see also [20, 24]) and on the real line in [15] (see, also, [22, 23]). In both cases the "truncation" produces processes which are more convenient and faster convergent than the corresponding "complete" ones.

Despite the simple expression of the rule, the major effort of the proposed approach depends on the construction of the coefficients $\{\mathcal{A}_k(t)\}_{k=1}^{j}$. We propose to compute them via the modified moments $\{M_k^{(s)}(t)\}_{k\in\mathbb{N}}$, $s = 0, 1, \dots, p$, with the kernels $K_s(x,t) = \frac{1}{(x-t)^{s+1}}$. By this way, the simultaneous approximation of $\mathcal{H}_0(fu_\gamma, t), \mathcal{H}_1(fu_\gamma, t), \dots, \mathcal{H}_p(fu_\gamma, t)$ can be easily performed with a computational cost of the same order.

The plan of the paper is the following: the next section contains some preliminary results and notation. In Section 3 we state the product rule with some results on stability and the rate of convergence of the error. In Section 4 we provide some details about the computation of the product rule coefficients. In the successive Section 5 we propose some numerical experiments in order to show the efficiency of the rule and provide some comparisons with other methods available in the literature. Finally, in Section 6 the proofs of the main results are stated.

2. Basic definitions and properties. Throughout the paper the constant $\mathcal{C}$ will be used several times having different meanings in different formulas. Moreover, from now on we write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ in order to express the fact that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \dots$ and $\mathcal{C} = \mathcal{C}(a, b, \dots)$ to state that $\mathcal{C}$ depends on $a, b, \dots$ Moreover, if $A, B \geq 0$ are quantities depending on some parameters, then we write $A \sim B$ if there exists a constant $0 < \mathcal{C} \neq (A, B)$ such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

Finally, $\mathbb{P}_m$ will denote the space of algebraic polynomials of degree at most m.

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2.1. Function spaces. With $u_\gamma(x) = x^\gamma e^{-x/2}$, $\gamma \geq 0$, we denote by C_{u_γ} the following set of functions

$$C_{u_\gamma} = \begin{cases} \left\{ f \in C^0((0,+\infty)) : \lim_{\substack{x\to+\infty \\ x\to 0^+}} (fu_\gamma)(x) = 0 \right\}, & \gamma > 0, \\ \left\{ f \in C^0([0,+\infty)) : \lim_{x\to+\infty} (fu_\gamma)(x) = 0 \right\}, & \gamma = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_{u_\gamma}} := \|fu_\gamma\| = \sup_{x\geq 0} |(fu_\gamma)(x)|,$$

where $C^0(E)$ is the space of continuous functions on the set E. In the sequel we use $\|f\|_E := \sup_{x\in E} |f(x)|$ for any subset $E \subset \mathbb{R}^+$.

For smoother functions we introduce the Sobolev-type spaces of order $r \in \mathbb{N}$,

$$W_r(u_\gamma) = \left\{ f \in C_{u_\gamma} \; : \; f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r u_\gamma\| < +\infty \right\},$$

where $AC((0,+\infty))$ denotes the set of all functions which are absolutely continuous on every closed subset of $(0,+\infty)$ and $\varphi(x) = \sqrt{x}$. We equip these spaces with the norm

$$\|f\|_{W_r(u_\gamma)} := \|fu_\gamma\| + \|f^{(r)}\varphi^r u_\gamma\|.$$

For any $f \in C_{u_\gamma}$ we consider the following main part of the k-th φ-modulus of smoothness (see [4] and also [17]):

$$\Omega_\varphi^k(f,t)_{u_\gamma} = \sup_{0<h\leq t} \|u_\gamma \Delta_{h\varphi}^k f\|_{I_{kh}},$$

where $I_{kh} = \left[4k^2h^2, \frac{\mathcal{C}}{h^2}\right]$, $\mathcal{C}$ is a fixed positive constant, and

$$\Delta_{h\varphi}^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\big(x + h\varphi(x)(k-i)\big).$$

The complete k-th modulus of smoothness is given by

$$\omega_\varphi^k(f,t)_{u_\gamma} = \Omega_\varphi^k(f,t)_{u_\gamma} + \inf_{P\in\mathbb{P}_{k-1}} \|(f-P)u_\gamma\|_{(0,4k^2t^2)} + \inf_{Q\in\mathbb{P}_{k-1}} \|(f-Q)u_\gamma\|_{(\frac{1}{t^2},+\infty)}.$$

By means of $\Omega_\varphi^k(f,t)_{u_\gamma}$ we define the Zygmund-type spaces

$$Z_\lambda(w_\alpha) := \left\{ f \in C_{u_\gamma} \; : \; \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{u_\gamma}}{t^\lambda} < +\infty \right\}$$

of parameter $0 < \lambda < k$, equipped with the norm

$$\|f\|_{Z_\lambda(u_\gamma)} = \|fu_\gamma\| + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{u_\gamma}}{t^\lambda}.$$

We recall that with $r = \lfloor \lambda \rfloor$, it is the case that $W_{r+1}(u_\gamma) \subseteq Z_\lambda(u_\gamma) \subseteq W_r(u_\gamma)$, and with $0 < \lambda < 1$ and $p \in \mathbb{N}$, $f^{(p)} \in Z_\lambda(u_\gamma\varphi^p)$ implies $f \in Z_{\lambda+p}(u_\gamma)$ and vice versa [7, Lemma 2.1]. Finally, it is useful to recall that for functions belonging to $Z_\lambda(u_\gamma), 0 < \lambda < 1$, it holds that

$$\omega_\varphi^r(f,t)_{u_\gamma\varphi^p} \sim \Omega_\varphi^r(f,t)_{u_\gamma\varphi^p} \tag{2.1}$$

(see [4, p. 189]).

2.2. Orthogonal polynomials and truncated Lagrange interpolation. Let the function $w_\alpha(x) = e^{-x}x^\alpha$ be the Laguerre weight of parameter $\alpha > -1$, and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients

$$p_m(w_\alpha, x) = \gamma_m(w_\alpha)x^m + \text{ terms of lower degree}, \qquad \gamma_m(w_\alpha) > 0.$$

Denoting by $x_{m,k}, k = 1, \ldots, m$, the zeros of $p_m(w_\alpha)$ in increasing order, we recall that (see [26])

$$\frac{\mathcal{C}(\alpha)}{m} < x_{m,1} < x_{m,2} < \ldots < x_{m,m} < 4m + 2\alpha - \mathcal{C}m^{\frac{1}{3}}.$$

From now on, for any fixed $0 < \theta < 1$, the integer $j := j(m)$ denotes the index of the zero of $p_m(w_\alpha)$ such that

$$x_{m,j} = \min_{k=1,2,..,m} \{x_{m,k} \, : \, x_{m,k} \geq 4m\theta\}. \tag{2.2}$$

Inside the segment $(0, x_{m,j})$ the distance between two consecutive zeros of $p_m(w_\alpha)$ can be estimated as follows

$$\Delta x_{m,k} \sim \Delta x_{m,k-1} \sim \sqrt{\frac{x_{m,k}}{m}}, \qquad \Delta x_{m,k} = x_{m,k+1} - x_{m,k}, \qquad k = 1, 2, \ldots, j.$$

Let $\mathcal{L}_{m+1}(w_\alpha, g)$ be the Lagrange polynomial interpolating a given function g at the zeros of $p_m(w_\alpha, x)(4m - x)$. Denote by χ_j the characteristic function of the segment $(0, x_{m,j})$ with j defined in (2.2). The Lagrange polynomial $L_{m+1}(w_\alpha, g) := \mathcal{L}_{m+1}(w_\alpha, g\chi_j)$ defined in [10] (see also [12]) can be expressed as

$$L_{m+1}(w_\alpha, g, x) = \sum_{k=1}^{j} l_{m,k}(x)\frac{4m - x}{4m - x_{m,k}} g(x_{m,k}) =: \sum_{k=1}^{j} \ell_{m,k}(x) g(x_{m,k}), \tag{2.3}$$

where $l_{m,k}(x) = \frac{p_m(w_\alpha, x)}{p'_m(w_\alpha, x_{m,k})(x - x_{m,k})}$. Setting

$$\mathcal{P}^*_m = \{p \in \mathbb{P}_m \, : \, p(x_{m,k}) = p(4m) = 0, \quad k > j\} \subset \mathbb{P}_m, \tag{2.4}$$

$L_{m+1}(w_\alpha, g)$ belongs to $\mathcal{P}^*_m$, and the operator $L_{m+1}(w_\alpha)$ projects C_u onto $\mathcal{P}^*_m$.

About the simultaneous approximation of a function and its derivatives, we recall the following result [7, Theorem 2.2]:

THEOREM 2.1. *If $f \in Z_{p+\lambda}(u_\gamma)$ with $0 < \lambda < 1$, $p \in \mathbb{N}$, and α, γ being two real parameters satisfying the inequality*

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}, \tag{2.5}$$

then, for any integer $0 \leq k \leq p$,

$$\|(f - L_{m+1}(w_\alpha, f))^{(k)} \varphi^k u_\gamma\| \leq \mathcal{C} \left\{ \frac{\log m}{(\sqrt{m})^{p+\lambda-k}} \|f\|_{Z_{p+\lambda}(u_\gamma)} + e^{-Am} \|f u_\gamma\| \right\},$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

REMARK 2.2. In particular, if $f \in W_{p+r}(u_\gamma)$ with $r \geq 1$, we have

$$\|(f - L_{m+1}(w_\alpha, f))^{(k)} \varphi^k u_\gamma\| \leq \mathcal{C} \left\{ \frac{\log m}{(\sqrt{m})^{p+r-k}} \|f\|_{W_{p+r}(u_\gamma)} + e^{-Am} \|f u_\gamma\| \right\},$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

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3. The method. With $u_\gamma(x) = e^{-\frac{x}{2}} x^\gamma, \gamma \geq 0$, consider

$$\mathcal{H}_p(fu_\gamma, t) := \int_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}} u_\gamma(x) dx, \tag{3.1}$$

where p is a nonnegative integer, $t > 0$, and the integral is defined in the Hadamard sense.

Following an argument used in [8], the existence of the right-hand side in (3.1) for $f^{(p)}$ satisfying a Dini-type condition is assured by the following theorem:

THEOREM 3.1. *Let $p \geq 1$, $\gamma \geq 0$. For any function f such that*

$$\int_0^1 \frac{\Omega_\varphi(f^{(p)}, t)_{u_\gamma \varphi^p}}{t} dt < +\infty$$

and for any $t > 0$, it holds that

$$t^p |\mathcal{H}_p(fu_\gamma, t)| \leq \mathcal{C} \left(\int_0^1 \frac{\Omega_\varphi(f^{(p)}, t)_{u_\gamma \varphi^p}}{t} dt + \|f\|_{W_p(u_\gamma)} \right), \quad 0 < \mathcal{C} \neq \mathcal{C}(f, t). \tag{3.2}$$

REMARK 3.2. In particular, if $f^{(p)} \in Z_\lambda(u_\gamma \varphi^p)$, then by (3.2) we deduce that

$$t^p |\mathcal{H}_p(fu_\gamma, t)| \leq \mathcal{C} \left(\|f\|_{Z_{p+\lambda}(u_\gamma)} + \|f\|_{W_p(u_\gamma)} \right), \qquad 0 < \mathcal{C} \neq \mathcal{C}(f, t),$$

and the function $\mathcal{H}_p(fu_\gamma, t)$ has an algebraic singularity of order p at zero.

Now we introduce a product integration rule based on the Lagrange interpolation polynomial $L_{m+1}(w_\alpha, f)$ defined in (2.3). Indeed, replacing f in (3.1) by $L_{m+1}(w_\alpha, f)$, we have

$$\mathcal{H}_p(fu_\gamma, t) = \mathcal{H}_{p,m}(fu_\gamma, t) + e_{p,m}(fu_\gamma, t), \tag{3.3}$$

where, taking into account (2.3),

$$\mathcal{H}_{p,m}(fu_\gamma, t) = \mathcal{H}_p(L_{m+1}(w_\alpha, f) u_\gamma, t) := \sum_{k=1}^{j} f(x_{m,k}) \mathcal{A}_k(t),$$
$$\mathcal{A}_k(t) = \int_0^{+\infty} \frac{\ell_{m,k}(x)}{(x-t)^{p+1}} u_\gamma(x) dx,$$

and

$$e_{p,m}(fu_\gamma, t) = \mathcal{H}_p(fu_\gamma, t) - \mathcal{H}_{p,m}(fu_\gamma, t) = \mathcal{H}_p\big((f - L_{m+1}(w_\alpha, f)) u_\gamma, t\big). \tag{3.4}$$

By definition of $L_{m+1}(w_\alpha, f)$ it follows that

$$e_{p,m}(fu_\gamma, t) = 0, \qquad \forall f \in \mathcal{P}_m^*$$

with $\mathcal{P}_m^*$ defined in (2.4). In Section 4 we give some details about the effective computation of the coefficients $\{\mathcal{A}_k(t)\}_{k=1}^{j}$. For now we state some results about the stability and convergence of (3.3) in some suitable subspaces of C_{u_γ}.

THEOREM 3.3. *For any $t > 0$, for $f \in Z_{p+\lambda}(u_\gamma)$ with $0 < \lambda < 1$, and if α, γ satisfy the condition*

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}, \tag{3.5}$$

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then it holds that

$$t^p|\mathcal{H}_{p,m}(fu_\gamma,t)| \le \mathcal{C}\left(\|f\|_{Z_{p+\lambda}(u_\gamma)} + \|f\|_{W_p(u_\gamma)}\right)\log m,$$

where $0<\mathcal{C}\neq\mathcal{C}(m,f,t)$.

By the previous theorem it follows that for any $t>0$ the product rule is stable except for the extra factor $\log m$. A result concerning the error estimate is given in the next theorem.

THEOREM 3.4. *Let* $0<\lambda<1$. *For any* $t>0$, *if* $f\in Z_{\lambda+p+q}(u_\gamma)$, $q\in\mathbb{R}^+$, *and* α,γ *satisfying* (3.5), *then*

$$t^p|e_{p,m}(fu_\gamma,t)| \le \mathcal{C}\frac{\|f\|_{Z_{\lambda+p+q}(u_\gamma)}}{\sqrt{m^{\lambda+q}}}\log m, \tag{3.6}$$

and if $f\in W_{p+r}(u_\gamma)$,$r\in\mathbb{N}, r\ge 1$, *then we obtain*

$$t^p|e_{p,m}(fu_\gamma,t)| \le \mathcal{C}\frac{\|f\|_{W_{p+r}(u_\gamma)}}{\sqrt{m^{r}}}\log m, \tag{3.7}$$

where $0<\mathcal{C}\neq\mathcal{C}(m,f,t)$.

REMARK 3.5. By the estimates (3.6) and (3.7) it follows that the error of the product rule behaves like the best approximation error of the function space f belongs to, except for the extra factor $\log m$.

4. Computational details. In this section we provide some details for computing the coefficients $\{\mathcal{A}\}_{k=1}^j$ in (3.3). By the following expression of the fundamental Lagrange polynomials

$$l_{m,k}(w_\alpha,x) = \lambda_{m,k}(w_\alpha)\sum_{i=0}^{m-1} p_i(w_\alpha,x)p_i(w_\alpha,x_{m,k}),$$

it follows that

$$\{\mathcal{A}_k\}_{k=1}^j = \frac{\lambda_{m,k}(w_\alpha)}{4m-x_{m,k}}\sum_{i=0}^{m-1} p_i(w_\alpha,x_{m,k})\mathcal{H}_p\big((4m-\cdot)p_i(w_\alpha)u_\gamma,t\big).$$

Recalling the following three-term recurrence relation for Laguerre polynomials

$$\begin{cases} p_{-1}(w_\alpha,x)=0, \qquad p_0(w_\alpha,x)=\dfrac{1}{\sqrt{\Gamma(\alpha+1)}}, \\ a_{i+1}p_{i+1}(w_\alpha,x) = (x-b_i)p_i(w_\alpha,x) - a_ip_{i-1}(w_\alpha,x), \\ a_i=\sqrt{i(i+\alpha)}, \qquad b_i = 2i+\alpha+1, \end{cases} \tag{4.1}$$

and setting

$$M_i^{(p)}(t) := \int_0^{+\infty} \frac{p_i(w_\alpha,x)}{(x-t)^{p+1}}u_\gamma(x)dx, \qquad i=0,1,\dots,m,$$

we have

$$\mathcal{A}_k(t) = \frac{\lambda_{m,k}(w_\alpha)}{4m-x_{m,k}}\sum_{i=0}^{m-1} p_i(w_\alpha,x_{m,k})\left((4m-b_i)M_i^{(p)}(t) - a_{i+1}M_{i+1}^{(p)}(t) - a_iM_{i-1}^{(p)}(t)\right).$$

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Thus, we have to compute the sequence $\{M_i^{(p)}(t)\}_{i=0}^m$.

By (4.1), the following recursion scheme can be deduced

$$(4.2)\quad \begin{cases} M_1^{(0)}(t) = \frac{1}{a_1}\left\{d_0 + (t-b_0)M_0^{(0)}(t)\right\}, & \\ M_1^{(p)}(t) = \frac{1}{a_1}\left\{\frac{1}{\sqrt{\Gamma(\alpha+1)}}\mathcal{H}_p(u_{\gamma+1},t) - b_0 M_0^{(p)}(t)\right\}, & p \geq 1, \\ M_{i+1}^{(0)}(t) = \frac{1}{a_{i+1}}\left\{d_i + (t-b_i)M_i^{(0)}(t) - a_i M_{i-1}^{(0)}(t)\right\}, & i \geq 0, \\ M_{i+1}^{(p)}(t) = \frac{1}{a_{i+1}}\left\{M_i^{(p-1)}(t) + (t-b_i)M_i^{(p)}(t) - a_i M_{i-1}^{(p)}(t)\right\}, & p \geq 1, \\ d_i = \int_0^\infty p_i(w_\alpha,x)u_\gamma(x)dx, \quad i \geq 0. & \end{cases}$$

For computing $\{d_i\}_{i=0}^m$ we use the Gauss-Laguerre rule with respect to u_γ.

For the starting moments $M_0^{(p)}(t) = \mathcal{H}_p(u_\gamma,t)/\sqrt{\Gamma(\alpha+1)}$, for all $p \geq 0$, we use

$$\mathcal{H}_p(u_\gamma,t) = \frac{1}{p!}\frac{d^p}{dt^p}\mathcal{H}_0(u_\gamma,t),$$

with

$$\mathcal{H}_0(u_\gamma,t) = \begin{cases} -e^{-t/2}\mathrm{Ei}(t/2), & \gamma = 0, \\ -\pi t^\gamma e^{-t/2}\cot((1+\gamma)\pi) + 2^\gamma\Gamma(\gamma)\,{}_1F_1(1,1-\gamma,-t/2), & \gamma \neq 0,1,\dots, \end{cases}$$

where $\mathrm{Ei}(t)$ is the Exponential Integral function and ${}_1F_1(a,b,x)$ is the Confluent Hypergeometric function with their first derivatives given by

$$\frac{d}{dt}\mathrm{Ei}(t) = -\frac{d}{dt}\int_{-t}^{+\infty}\frac{e^{-x}}{x}dx = \frac{e^t}{t}, \qquad \frac{d}{dt}\,{}_1F_1(a,b;t) = \frac{a}{b}\,{}_1F_1(a+1,b+1,t).$$

As we can see, the computation of the $\{M_i^{(p)}(t)\}_{i=0}^m$ does not involve any derivative except the starting moments. Moreover, in order to compute $\{M_i^{(p)}(t)\}_{i=0}^m$, all the sequences $\{M_i^{(p-1)}(t), M_i^{(p-2)}(t), \dots, M_i^{(1)}(t), M_i^{(0)}(t)\}_{i=0}^m$ have to be determined, requiring a computational cost of about $5m^2 + 3mq(p+1)$ multiplicative operations, where q is the number of values of t. Since all the p sequences of the modified moments have to be computed, the simultaneous approximation of $\mathcal{H}_0(fu_\gamma,t), \mathcal{H}_1(fu_\gamma,t), \dots, \mathcal{H}_p(fu_\gamma,t)$ can be easily arranged with an additional computational cost of only $(p+1)m^2$ more multiplicative operations.

The sensitivity of the recurrence relation in (4.2) has been tested computationally for different values of α and γ,

$$\mathcal{M}_1(t) := \max_{0\leq i\leq 500}\left|\frac{M_{i,Q}^{(1)}(t) - M_{i,D}^{(1)}(t)}{M_{i,Q}^{(1)}(t)}\right|,$$

where $\{M_{i,D}^{(1)}(t)\}_{i=0}^{500}$ and $\{M_{i,Q}^{(1)}(t)\}_{i=0}^{500}$ are the sequences in double (eps_D) and quadruple (eps_Q) machine precision, respectively.

In Table 4.1 we present results only for the case $\alpha = 0.5, \gamma = 0.6$ since for other choices of α, γ they are similar. Looking at the table we can deduce that the recurrence relation (4.2) is essentially stable.

TABLE 4.1
$\mathcal{M}_1(t)$ *for different values of* t.

t	10^{-12}	10^{-9}	10^{-7}	10^{-3}	1	10	50
$\mathcal{M}_1(t)$	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	2.6e-15

5. Numerical tests. Now we propose some numerical tests to show the performance of the product rule (3.3). The density functions will be of different smoothness, and the numerical results will confirm the theoretical estimates. In the first two tests we also compare our results in the cases $p = 0$ and $p = 1$ with those obtained by other methods available in the literature. In the tables we will denote by [DBO2017a] and [DBO2017b] the methods recently proposed in [7] and [6], respectively. Moreover, we will denote by [IT1979] a proper variant of the method proposed by Ioakimidis and Theocaris in [9] (see also [18]) and already discussed in [7, p. 138]. Finally, in Example 5.2, we also report the results obtained by using the method [AD2008] introduced by Aimi and Diligenti in [1], which essentially reduces the problem to the interval $[0, 1]$ by a nonlinear transformation.

In all the examples, since the exact values of the integrals are unknown, we retain as exact the values computed with $m = 1000$, and we set

$$\bar{e}_{p,m}(fu_\gamma, t) = |\mathcal{H}_{p,m}(fu_\gamma, t) - \mathcal{H}_{p,1000}(fu_\gamma, t)|.$$

All computations have been performed in double-machine precision ($eps_D \sim 2.22044\text{e-}16$), and in the tables the symbol "$-$" means that the machine accuracy has been achieved.

Moreover, we use the following definition of the truncation index (see [3, p. 781])

$$j = \min_{k=1,\ldots,m} \{k : \lambda_{m,k} < eps_D\}, \tag{5.1}$$

taking into account that $\lambda_{m,k} \sim \Delta x_{m,k} w_\alpha(x_{m,k})$. The above definition is equivalent to (2.2) in the sense that there exists a $\theta \in (0, 1)$ such that $x_{m,j-1} < 4m\theta < x_{m,j}$, where j is defined in (5.1). To have an idea of the percentage of knots involved in the truncation process, depending on the choice of θ, see [21].

EXAMPLE 5.1. Let us consider the integral

$$\mathcal{H}_1(fu_\gamma, t) = \int\!\!\!\!\!\!= _0^{+\infty} \frac{\sin(x+5)}{(x-t)} x^{0.6} e^{-x} dx,$$

where $\gamma = 0.6$, $p = 0$, and $f(x) = \sin(x+5)e^{-x/2}$. We have $f \in W_r(u_\gamma)$ for all r, thus, according to (3.7), by choosing $\alpha = 0$ we expect fast convergence. This is confirmed by the numerical results shown in Table 5.1. In fact, with $j = 36$ evaluations of the function f, we get approximations of the integral with machine precision at different points t. As one can see in the Tables 5.2 and 5.3, the methods [DBO2017a] and [DBO2017b] also give satisfactory results, but, as already highlighted, both of them require the computations of the derivatives of the function f. In addition, the method [DBO2017a] is more expensive. In fact, it requires $j + q = 104$ evaluations of the density function. Finally, by inspecting Table 5.4, the rule [IT1979] appears to be slower for t approaching 0.

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TABLE 5.1
Errors $\bar{e}_{0,m}(f,t)$ by the present product rule in Example 5.1

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
10	10	7.9e-4	2.2e-3	2.5e-3	2.6e-3
20	18	2.0e-6	1.9e-5	5.2e-5	4.8e-5
30	23	7.1e-9	1.3e-7	5.0e-7	1.0e-6
40	27	9.8e-11	7.6e-11	1.0e-9	9.0e-10
50	30	2.3e-12	6.7e-12	2.2e-11	3.1e-11
60	33	3.7e-14	1.2e-13	2.8e-13	3.5e-13
70	36	–	–	–	–

TABLE 5.2
Errors by the method [DBO2017a] in Example 5.1.

m	j	q	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
20	20	19	1.8e-4	2.8e-5	1.3e-5	2.3e-5
40	35	28	1.4e-7	3.0e-8	8.2e-10	4.5e-10
80	53	40	8.1e-14	1.6e-14	9.0e-15	2.3e-14
100	59	45	–	–	–	–

TABLE 5.3
Errors by the method [DBO2017b] in Example 5.1.

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
6	6	1.1e-5	1.2e-5	1.6e-5	3.8e-5
13	13	2.1e-9	2.1e-9	3.7e-10	5.2e-10
22	19	–	–	–	–

TABLE 5.4
Errors by the method [IT1979] in Example 5.1.

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
10	10	4.1e-4	9.3e-5	1.8e-7	1.2e-9
40	26	2.7e-5	5.6e-7	1.8e-13	–
80	37	4.9e-6	1.1e-8	–	–

EXAMPLE 5.2. We consider

$$\mathcal{H}_1(fu_\gamma, t) = \oint_0^{+\infty} \frac{\sin(x+5)}{(x-t)^2} x^{0.6} e^{-x} dx,$$

where γ and f are the same as in the previous example and $p = 1$. Inspecting the results presented in Tables 5.5–5.8, considerations similar to the ones done in the previous example hold true both for the proposed product rule and for the rules [DBO2017a], [DBO2017b], and [IT1979]. In this case ($p = 1$) we are able to compare our results also with those obtained by the method [AD2008], and, as one can see in Table 5.9, the latter method is much slower than our product rule.

TABLE 5.5
Errors $\bar{e}_{1,m}(f,t)$ by the present product rule in Example 5.2.

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
10	10	4.1e-2	1.8e-2	1.1e-2	4.5e-3
20	19	5.7e-4	6.9e-5	7.2e-5	2.2e-4
30	23	9.1e-6	4.3e-6	2.0e-6	4.0e-7
40	27	3.2e-8	1.8e-8	8.8e-9	2.9e-9
50	31	4.9e-10	2.7e-10	1.5e-10	1.5e-19
60	34	6.1e-12	2.5e-12	1.4e-12	1.5e-12
70	36	1.2e-14	2.7e-15	5.1e-15	1.0e-15
80	39	–	–	–	–

TABLE 5.6
Errors by the method [DBO2017a] in Example 5.2.

m	j	q	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
20	20	19	2.5e-3	1.1e-3	1.2e-4	9.6e-5
40	35	28	3.9e-6	5.0e-7	1.2e-7	1.2e-7
80	53	40	5.2e-12	5.8e-13	4.8e-14	6.4e-14
100	59	45	5.4e-15	–	–	–

TABLE 5.7
Errors by the method in [DBO2017b] in Example 5.2.

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
5	5	4.8e-6	4.5e-6	9.8e-6	5.1e-6
10	10	1.0e-10	1.0e-10	3.5e-11	3.7e-11
20	18	eps	eps	eps	eps

TABLE 5.8
Errors by the method [IT1979] in Example 5.2.

m	j	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
10	10	8.3988e-4	1.7563e-4	9.5187e-8	2.9087e-11
40	26	4.9043e-5	8.8342e-7	9.0383e-14	–
80	37	8.4866e-6	1.7470e-8	–	–

TABLE 5.9
Errors by the method [AD2008] in Example 5.2.

m	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 5$
100	4.0e-3	1.4e-5,	2.7e-7	1.9e-9
200	3.7e-4	1.0e-6	2.2e-8	8.0e-11
300	2.6e-5	1.2e-7	1.5e-9	8.7e-12
400	9.9e-6	2.8e-8	8.7e-11	4.7e-12
500	1.4e-5	5.3e-8	5.4e-10	5.4e-12
600	8.6e-6	4.8e-8	5.7e-10	5.8e-12
700	5.2e-8	3.7e-8	3.0e-10	5.6e-12
800	5.7e-7	2.6e-8	8.7e-11	5.1e-12
900	1.6e-6	1.7e-8	3.6e-11	4.4e-12
1000	1.5e-6	1.0e-8	6.4e-11	3.6e-12

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EXAMPLE 5.3. Let

$$\mathcal{H}_1(fu_\gamma, t) = \oint_0^{+\infty} \frac{x^{5/4}}{(4+x^2)^4(x-t)^2} dx,$$

where $\gamma = 1.25$, $f(x) = \frac{e^{x/2}}{(4+x^2)^4}$, and $p = 1$. Taking into account that $f \in W_{13}(u_\gamma)$, the theoretical error behaves like $\log m/m^6$. According to (2.5) we apply our rule choosing

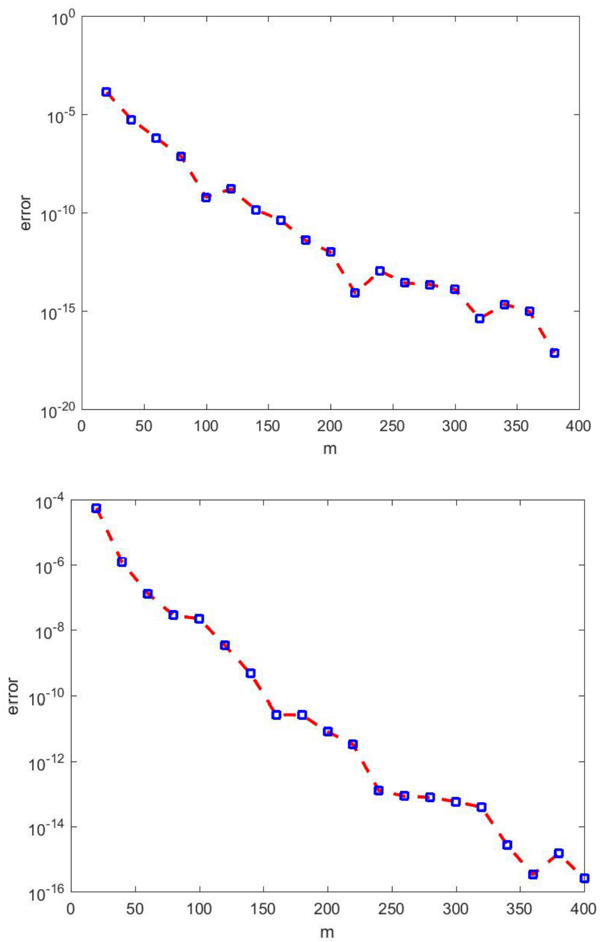

FIG. 5.1. *Errors* $\bar{e}_{1,m}(f, 0.001)$ *(top) and* $\bar{e}_{1,m}(f, 5)$ *(bottom) in Example 5.3.*

$\alpha = 0.5$. In Figures 5.1 and 5.2 we display the behaviour of the errors $\bar{e}_{1,m}(f,t)$ for increasing values of m at different points t. As one can see, the numerical errors are in agreement with the theoretical estimate. In fact for $m = 400$ ($j = 243$ evaluations of the function f) we obtain 15 exact digits.

EXAMPLE 5.4. Finally, we take

$$\mathcal{H}_1(fu_\gamma, t) = \oint_0^{+\infty} \frac{|x-2|^{11/2}}{(x^2+5)^2(x-t)^2} x^{5/2} e^{-x/2} dx,$$

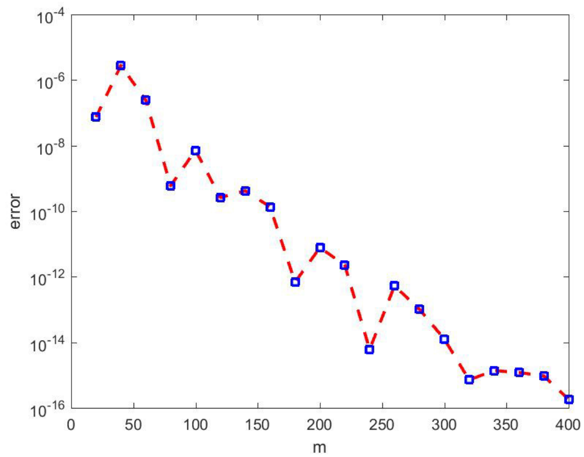

FIG. 5.2. *Errors* $\bar{e}_{1,m}(f, 10)$ *in Example 5.3.*

TABLE 5.10
Errors $\bar{e}_{1,m}(f, t)$ *in Example 5.4.*

m	j	$t = 0.5$	$t = 2.02$	$t = 4.1$
100	66	1.8e-4	1.8e-4	3.2e-5
200	96	1.3e-5	6.3e-5	3.4e-5
400	137	1.7e-7	2.6e-6	3.6e-6
600	169	3.7e-8	4.5e-7	1.8e-6
700	134	2.5e-7	8.9e-7	4.7e-7
800	196	1.5e-7	2.1e-6	1.3e-7

where $\gamma = 2.5$, $f(x) = \frac{|x-2|^{11/2}}{(x^2+5)^2}$, and $p = 1$. In this case $f \in W_5(u_\gamma)$, and, according to the theoretical estimate, we expect slower convergence $\big(\mathcal{O}(\log m/m^2)\big)$. We apply our rule choosing $\alpha = 2.6$. In Table 5.10 we present the results obtained for $t = 0.5, 2.02, 4.1$, and in Figure 5.3 we display the behaviour of the numerical error $\bar{e}_{1,900}(f, t)$ for $t \in [0, 4]$. In agreement with the theoretical expectation, the numerical errors increase as t approaches the critical point 2 of the density function f.

6. The proofs. First we recall the following Lemma [8, Lemma 2.1]

LEMMA 6.1. *Let* $f \in C_{u_\gamma}$ *and* $P_m \in \mathbb{P}_m$. *Then*

$$\int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi(f - P_m, t)_{u_\gamma}}{t} dt \le \mathcal{C} \left(\|(f - P_m)u_\gamma\| + \int_0^{\frac{1}{\sqrt{m}}} \frac{\omega_\varphi^r(f, t)_{u_\gamma}}{t} dt \right),$$

where $r \in \mathbb{N}$, *with* $r < m$ *and* $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

LEMMA 6.2. *For any* $f \in Z_{p+\lambda}(u_\gamma)$ *with* $0 < \lambda < 1$ *and* $p \ge 0$, *under the assumption*

$$\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}, \tag{6.1}$$

we have

$$\int_0^1 \frac{\Omega_\varphi(L_{m+1}^{(p)}(w_\alpha, f), t)_{u_\gamma\varphi^p}}{t} dt \le \mathcal{C} \left(\|f\|_{W_p(u_\gamma)} + \|f\|_{Z_{p+\lambda}(u_\gamma)} \right) \log m,$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

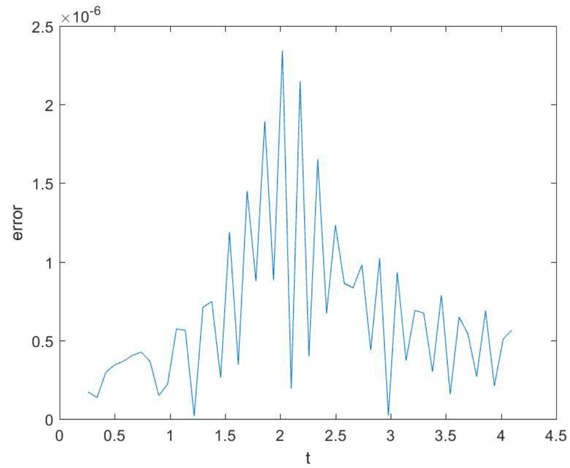

FIG. 5.3. *Errors* $\bar{e}_{1,900}(f,t)$ *for* $t \in [0,4]$ *in Example* 5.4.

Proof. We have

$$\int_0^1 \frac{\Omega_\varphi(L^{(p)}_{m+1}(w_\alpha,f),t)_{u_\gamma\varphi^p}}{t}dt \le \int_0^1 \frac{\Omega_\varphi((f-L_{m+1}(w_\alpha,f))^{(p)},t)_{u_\gamma\varphi^p}}{t}dt + \int_0^1 \frac{\Omega_\varphi(f^{(p)},t)_{u_\gamma\varphi^p}}{t}dt =: A_1 + A_2.$$

By Lemma 6.1 with $P = L_{m+1}(w_\alpha, f)$ and taking into account (2.1), it follows that

$$\begin{aligned} A_1 &\le \int_0^{1/\sqrt{m}} \frac{\Omega_\varphi((f-L_{m+1}(w_\alpha,f))^{(p)},t)_{u_\gamma\varphi^p}}{t}dt + \int_{1/\sqrt{m}}^1 \frac{\Omega_\varphi((f-L_{m+1}(w_\alpha,f))^{(p)},t)_{u_\gamma\varphi^p}}{t}dt \\ &\le \mathcal{C}\bigg(\|[f-L_{m+1}(w_\alpha,f)]^{(p)}\varphi^p u_\gamma\| + \int_0^{1/\sqrt{m}} \frac{\Omega^r_\varphi(f^{(p)},t)_{u_\gamma\varphi^p}}{t}dt + \|[f-L_{m+1}(w_\alpha,f)]^{(p)}\varphi^p u_\gamma\| \log m \bigg), \end{aligned}$$

and by Theorem 2.1 under assumption (6.1), we obtain

$$A_1 \le \mathcal{C}\frac{\log m}{(\sqrt{m})^\lambda}\|f\|_{Z_{p+\lambda}(u_\gamma)}. \tag{6.2}$$

Moreover,

$$\begin{aligned} A_2 &\le \int_0^{1/\sqrt{m}} \frac{\Omega_\varphi(f^{(p)},t)_{u_\gamma\varphi^p}}{t}dt + \int_{1/\sqrt{m}}^1 \frac{\Omega_\varphi(f^{(p)},t)_{u_\gamma\varphi^p}}{t}dt \\ &\le \mathcal{C}\|f\|_{Z_{p+\lambda}(u_\gamma)} + \mathcal{C}\|f\|_{W_p(u_\gamma)}\log m, \end{aligned}$$

and the lemma follows. ∎

Proof of Theorem 3.3. By Theorem 3.1

$$t^p|\mathcal{H}_{p,m}(fu_\gamma,t)| = t^p\,|\mathcal{H}_p(L_{m+1}(w_\alpha,f)u_\gamma,t)| \\ \leq \mathcal{C}\int_0^1 \frac{\Omega_\varphi(L^{(p)}_{m+1}(w_\alpha,f),t)_{u_\gamma\varphi^p}}{t}dt, +\|L_{m+1}(w_\alpha,f)\|_{W_p(u_\gamma)},$$

and by Lemma 6.2 and Theorem 2.1 under assumption (6.1), it follows that

$$t^p|\mathcal{H}_{p,m}(fu_\gamma,t)| \leq \mathcal{C}\left(\|f\|_{Z_{p+\lambda}(u_\gamma)} + \|f\|_{W_p(u_\gamma)}\right)\log m \\ +\|(f-L_{m+1}(w_\alpha,f))\|_{W_p(u_\gamma)} + \|f\|_{W_p(u_\gamma)} \\ \leq \mathcal{C}\left(\|f\|_{Z_{p+\lambda}(u_\gamma)} + \|f\|_{W_p(u_\gamma)}\right)\log m. \qquad \square$$

Proof of Theorem 3.4. Recalling (3.4) and using Theorem 3.1

$$t^p|e_{p,m}(fu_\gamma,t)| \leq \mathcal{C}\left(\int_0^1 \frac{\Omega_\varphi((f-L_{m+1}(w_\alpha,f))^{(p)},t)_{u_\gamma\varphi^p}}{t}dt \right. \\ \left. +\|f-L_{m+1}(w_\alpha,f)\|_{W_p(u_\gamma)}\right).$$

Now, by (6.2) and Theorem 2.1 under assumption (6.1), it follows that

$$t^p|e_{p,m}(fu_\gamma,t)| \leq \mathcal{C}\frac{\log m}{(\sqrt{m})^{\lambda+q}}\|f^{(p)}\|_{Z_{\lambda+q}(u_\gamma\varphi^p)},$$

i.e., the assertion. $\square$

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