ON THE MOMENTUM OF PSEUDOSTABLE POPULATIONS

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Abstract

In a pathbreaking paper Keyfitz (1971) calculated the demographic momentum, i.e. the amount of further population growth if an immediate reduction of fertility to bare replacement would occur. While he assumes a stable initial population, our research aims to extend it to so-called pseudostable populations where fertility declines at a constant rate while mortality remains fixed. We were able to analytically obtain interesting insights into the momentum of growth of such populations. Comparing theoretical results in a pseudostable setting with the projections according to the component method showed a fit that was un-expectedly remarkably good.

A peculiar effect of pseudostability is the monotonous decline of the momentum over time. Starting from a certain fixed level in the remote past, the momentum converges to zero for an infinite time horizon. Interestingly enough, there is a time where there is neither a positive nor a negative momentum. To put it numerically, a remarkable asymmetry in the momentum is revealed. While an immediate fertility change may lead to a maximal increase of the population by a factor between 2 and 3, the maximal decrease runs up to 100%.

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1 Introduction

The momentum of population growth is a classic example of an apparently counterintuitive effect and was introduced by Nathan Keyfitz about half a century ago (1971). Revisiting a discussion among French demographers from around the mid twentieth century (Bourgeois-Pichat, 1968; Vincent, 1945), Keyfitz' contribution opened up a new avenue of population research. The answer to a simple question — what would happen to the size of a population if fertility dropped instantaneously to the replacement level? — baffled many and some even refused to believe that Brazil's population would still grow by about two thirds in this scenario as the first author of the present paper vividly recalls from the 1974 World Population Conference in Bucharest. This amount of continued population growth after an instantaneous decline in the fertility rate to the replacement level is called *popula-tion momentum*. The scenario of an instantaneous surge in fertility to the replacement level would lead, analogously, to a population that continues to decline for several decades.

Such a sudden drop/surge in fertility is not possible in reality. Nevertheless, the resulting growth or decline can be considered as a conservative estimate: Previously growing populations will still grow by *at least* the amount specified by the momentum. Likewise, previously declining populations will shrink *at least* by the specified amount.

Whereas Keyfitz' analytic expression for estimating population momentum refers to stable populations, the concept of demographic momentum can be applied to arbitrary popu-lations. The present paper studies the momentum of growth for a special family of popula-tions, namely for *pseudostable* ones.

Stable populations are characterized by time-invariant age-specific fertility and mortality rates. Although such assumptions never prevail in reality over a longer period, the analytical tractability is a great advantage. There have been several attempts to relax these restrictive assumptions without giving up the whole flair of an analytic treatment. One early example of studying the effect of time-changing vitality rates to the growth and structure of populations was Coale's idea to include uniformly shrinking fertility rates (Coale and Zelnik, 1963; see also Chapt. 4 in Coale, 1972), later called pseudostable populations by Feichtinger and Vogelsang (1978). For a brief introduction into the field needed in our analysis of the pseudostable momentum see Section 3 below.

This paper studies the behavior of the time path of the momentum $M(t)$, which is defined as the ratio of the ultimate population size divided by its initial size. In particular, we de-rived a comprehensive, qualitative characterization of the momentum of population growth under pseudostable conditions. Our results show that $M(t)$ is a monotonously decreasing S-shaped function converging to zero for increasing reference times, but remarkably to a finite value given by the ratio e_0/μ in the remote past. This fact guarantees the existence of a time $\hat{t}$ with neither a positive nor a negative momentum, i.e. $M(\hat{t}) = 1$. Moreover, we present a dynamic sensitivity analysis with respect to the key parameter of pseudostability, namely the rate k of fertility decline.

Since the analytic approach to pseudostable populations relies on the simplification of a fixed unique birth age, it is important to compare the theoretical results with those obtained by the standard population projection method. Generally speaking, the fit of the analytical pseudostable approximations to the numerical projection turned out to be reasonably good.

As the formal treatment of pseudostable populations provides new insights our analysis extends existing knowledge in several ways, including:

A) It allows an analytical approach to populations late in the first demographic transition, when mortality declines have stalled but fertility continues to fall.

B) It can be instructive for historical demographers. As Coale (1972) already pointed out, the demographic situation in the United States during the first decades of the twentieth century resembles the pseudostable case.

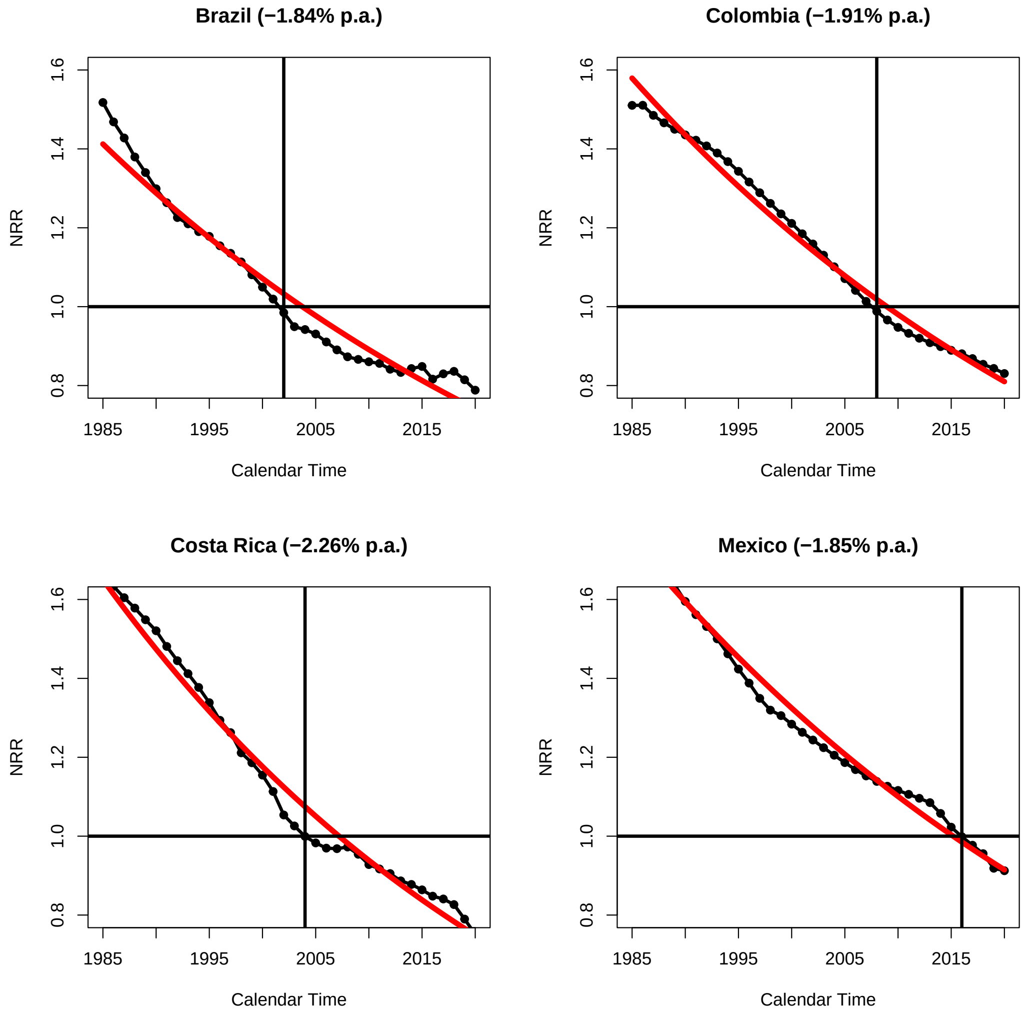

Figure 1: Net reproduction rate ("NRR"), exponential fit (red) and estimated annual changes (black) in percent in four selected Latin American countries between 1985 and 2020. Data source: Own illustration based on data from the United Nations (2022).

C) It can be employed for the analysis of contemporary populations. While fertility is declining in many parts of the world, Latin America is particularly interesting for two reasons:

1) Many of these countries fit the pseudostable scenario of proportional reductions in fertility extremely well, combined with relatively little variation in the age at childbearing.

2) The net reproduction rate has crossed the replacement level (NRR=1) in recent years, which is the time for which we identified the most interesting dynamics in a pseudostable setting. Figure 1 shows the development of the net reproduction rate in four selected Latin American countries for the years 1985–2020. The black lines denote the observed fertility. The red lines show the fit of the fertility decline assuming a constant decline. The titles in each of the four panels do not only name the countries but also report the estimated declines in the net reproduction rate per year. We selected Colombia from these four countries to illustrate our results. Our analytical treatment can then be used as a guide for interpretating empirically observed results (as suggested by Lotka, 1938). Likewise they can be used to forecast the age structure, the population size, the birth trajectory as well as the population momentum.

The paper is organized as follows. The following two sections sketch the population momentum in Section 2 as well as the pseudostable framework in Section 3. After bringing together both issues, Section 4 presents the main results on the pseudostable momentum. Section 5 contains conclusions and ideas for possible extensions.

Appendix A summarizes some important facts about the pseudostable populations. Proofs of propositions are given in appendix B, and additional material is in appendix C.

2 On the Momentum of Population Growth

Following the seminal paper by Keyfitz (1971; see also Keyfitz, 1977), we start with a stable age structure with growth rate r and birth rate b; for a more detailed explanation, we refer to Appendix A. Denoting life expectancy at birth by e_0, the mean age of childbearing in the stationary population by μ, and the net reproduction rate (NRR) of the initial population by R_0, reveals an immediate drop in fertility to replacement level, i.e. to an NRR of 1 the following expression for the momentum:

$$M = \left(\frac{e_0}{\mu}\right)\left(\frac{b}{r}\right)\frac{R_0 - 1}{R_0} \tag{1}$$

The momentum expresses the ratio of the resulting ultimate total stationary population divided by the size of the initial population before the intended change in fertility. Note that (1) also remains valid for negative population growth. Hence, the initial stable growth rate r is below zero corresponding to NRR$= \exp(rT) < 1$, where T denotes the mean length of a generation. A decreasing population continues to shrink for a while until the stationary level will be reached.

The proof of Eq. (1) uses Fisher's (1930) "reproductive value", measuring the average number of future children a woman of a certain age will have during her remaining reproductive pe-riod. According to standard, we consider a one-sex model that restricts us to the female component of the population.

As mentioned, the concept of the population momentum remains valid for any population, not only for those whose age distribution is stable as in Keyfitz' (1971) analytic formula. Preston and Guillot (1997) introduced an equation expressing population momentum for *any* age structure (see also Preston et al., 2001, p.162):

$$M(t) = \int_0^{\omega} \frac{c(a,t)}{c_s(a)} w(a)\mathrm{d}a, \tag{2}$$

where $c(a,t)$ is the proportionate age distribution of the population at time t when replacement-level fertility is imposed. The age distribution in the denominator of Eq. (2) is the stationary one that will eventually emerge after replacement-level fertility has been in place for many years. The third quantity in Eq. (2), $w(a)$ is given as

$$w(a) = \frac{l(a)v(a)}{\mu} \tag{3}$$

where

$$v(a) = \frac{1}{l(a)} \int_a^{\omega} l(x)m^*(x)\mathrm{d}x \tag{4}$$

denotes the reproductive value of an a year old woman, as explained above with the replacement fertility rates $m^*(x)$.

A characteristic feature of Coale's approach is to assume that all births occur at a unique age, namely μ. Starting with the time when replacement-level fertility is imposed, each female expects exactly one birth at age μ, none below and none above.

Thus, we have

$$v(a) = \begin{cases} 1 & \text{for} \quad a \leq \mu \\ 0 & \text{for} \quad a > \mu \end{cases} \tag{5}$$

Now from Eq. 2, we obtain

$$M(t) = \frac{e_0}{\mu} \int_0^{\mu} \frac{c(a,t)}{l(a)} \mathrm{d}a. \tag{6}$$

For an overview of the demographic momentum, we refer to Preston et al. (2001, chap. 7.7) and Schoen (2007, chap. 4). While Keyfitz (1971) assumes an immediate change of fertility rates to replacement level, Li and Tuljapurkar (1999; 2000) and Goldstein (2002) deal with gradual changes of vitality rates to a stationary situation.

3 Fundamentals of pseudostable populations

The assumptions that underly the stable population model allow for a mathematical treatment of the transient and long-term dynamics of such a population. Relaxing those assumptions is a step towards a more realistic perspective. In Chapter 4 of his groundbreaking book, Coale (1972) considered the case of fertility declining at a constant rate. This added flexibil-ity allows us to investigate the demographic consequences of the fertility decline observed in many countries during the demographic transition in a succinct, analytical way. More specifically, the pseudostable approach answers the question of how a long-term fertility decline at a fixed annual rate influences the age distribution as well as related demographic indices.

Coale (1972, Chapter 4) studied a particular instance of changing fertility schedules: Assuming constant mortality and fertility subject to a constant annual change with a fixed age pattern, he was able to derive analytic expressions for the underlying population decline in this scenario of a fertility decline (see also Coale and Zelnik (1963); Feichtinger and Vogelsang (1978); Kim and Schoen (1996); Schoen (2007); for earlier and related work). Remarkably, the approach pioneered by Coale and Zelnik (1963) provides a bouquet of analytic results that are different but comparable to those in a stable framework. This fact motivated Feichtinger and Vogelsang (1978) to denominate the Coale-Zelnik populations as pseudostable. Next, we summarize some of our findings that are crucial for understanding the growth and the structure of pseudostable populations.

Consider a one-sex population dynamic model in continuous time with age specific fertility rates in period

$$m(a,t) = m(a,0)\exp(kt) \text{ for } \alpha \leq a \leq \beta \tag{7}$$

in period t at age a, where $k < 0$ measures the annual rate of fertility decline. Note that fertility has a fixed age structure, but with changing level at a constant rate. Then for the time-dependent net reproduction rate, $R(t) = \int_{\alpha}^{\beta} l(a)m(a,t)\mathrm{d}a$ holds:

$$R(t) = R(0)\exp(kt), \qquad \mathrm{d}\log R(t)/\mathrm{d}t = k. \tag{8}$$

To simplify the analysis we choose the time $t = 0$ such that there prevails exact replacement ($R(0) = 1$). Figure 2 illustrates the time path of an exponentially shrinking net reproduction rate with $k = -2\%$, i.e., about the pace observed in the four Latin American countries shown in Fig. 1. While the next sections show asymptotic behavior, we can already see that a pseudostable population describes a transient phase: With an empirically observed pace of fertility decline of about two percent annually, it would only take about one hundred years from the current highest observed net reproduction rate (2.937 in Niger) to the

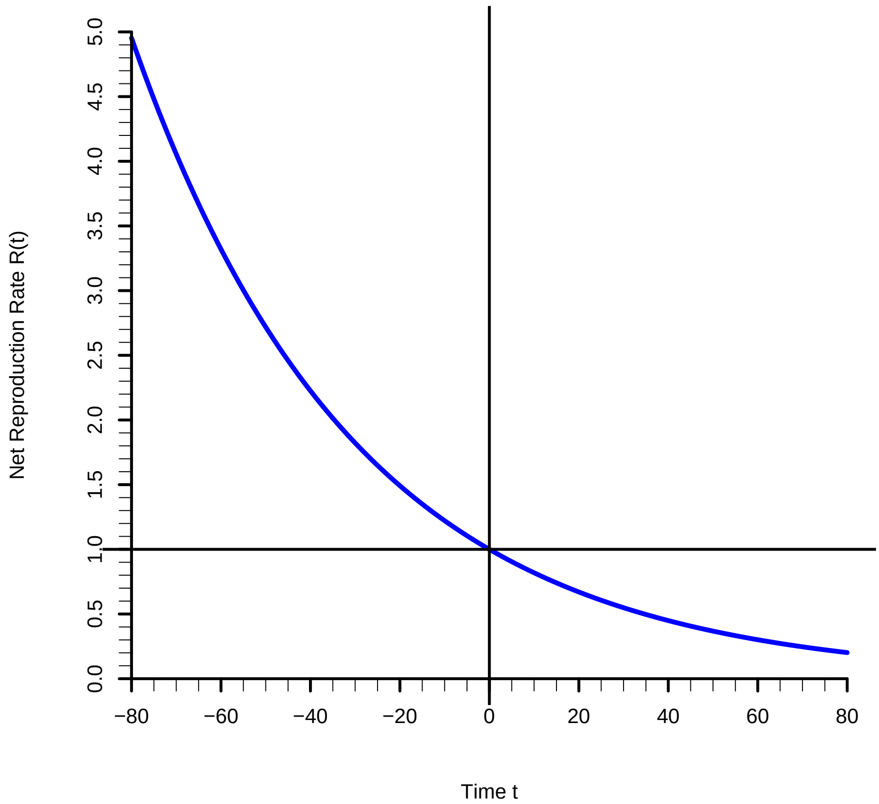

Figure 2: Net reproduction rate $R(t)$, assuming a constant rate of fertility decline of two percent. Time axis $t = 0$ is fixed when $R(t) = 1$.

lowest (0.358 in Hongkong, see United Nations (2022) for both NRR estimates).[1]

Our first goal is to find out the birth trajectory. Inserting assumption 7 in the renewal equation for births $B(t)$, we get

$$B(t) = \int_{\alpha}^{\beta} B(t-a)l(a)m(a,t)\mathrm{d}a = R(0)\exp(kt)\int_{\alpha}^{\beta} B(t-a)l(a)m(a,0)\mathrm{d}a. \quad (9)$$

Assuming a unique age at which females give birth, denoted as $\mu(t)$, and applying the mean value theorem of integral calculus, yields

[1] $\log(0.358/2.937)/(-0.02) = 105.2305$

$$B(t) = B(t - \mu(t))R(t). \quad (10)$$

To solve (10), Coale (1972, p.119) assumed that $\mu(t)$ is a fixed number,

$$\mu = \frac{\int_\alpha^\beta a l(a) m(a) \mathrm{d}a}{\int_\alpha^\beta l(a) m(a) \mathrm{d}a}$$

i.e., the mean age of childbearing in the stationary population.

Then, (10) delivers a linear difference equation for $\log B(t)$.[2] Solving it by a quadratic approach leads to the following birth sequence

$$B(t) = B(0) \exp\left[\frac{kt}{2} + \frac{kt^2}{2\mu}\right]. \quad (11)$$

Note that:

- With known k, t, and μ, $B(0)$ uniquely determines the birth trajectory.
- Note also the extension from geometric growth for stable populations to a quadratic exponential function in the pseudostable case.
- $B(t)$ is a symmetric bell-shaped curve reaching its maximum at $t = -\mu/2$. This might seem counterintuitive for two reasons: 1) We observe an increase in births despite a continuously declining level of fertility (until $t = -\mu/2$). 2) The number of births declines despite $NRR > 1$ for $-\mu/2 < t < 0$. This can be understood by seeing the number of births as the scalar product of the (vector of) age-specific fertility rates and the (vector of) corresponding number of women at fertile ages where the former is continually falling but the latter is still increasing.

Figure 3 illustrates the dependence of birth trajectories on k, i.e. the rate of fertility decline, and the mean age at childbearing μ. The upper panel confirms — most visibly for the relatively fast annual fertility decline of 5% ($k = -0.05$) in bright blue — that we have a bell

[2]See Appendix A for details.

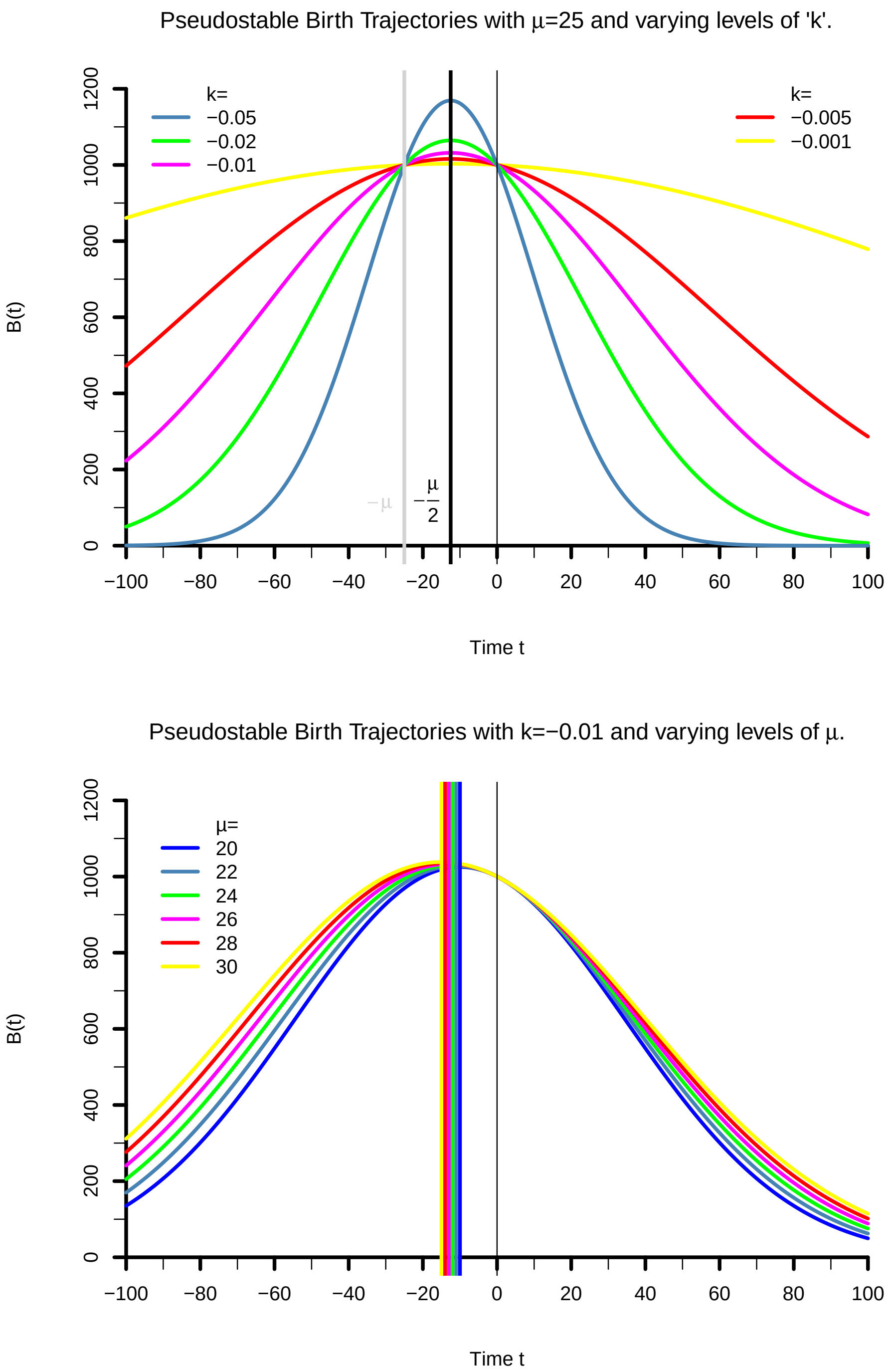

Figure 3: Analytical pseudostable birth trajectories.

shaped curve. The larger the absolute value $|k|$ is, the steeper the $B(t)$; small $|k|$ lead to 'flat' trajectories being less concentrated around $-\mu/2$; the peak in births is reached exactly at $t = -\mu/2$. Due to the quadratic shape the same number of births are observed at $t = -\mu$ as in $t = 0$. The lower panel shows how the timing of the peak in births depends on μ for a given pace of fertility decline (1% annually), but not the level or shape. Thus, k determines the shape and scale of the birth trajectory, whereas μ is the location parameter.

It is astounding that the strong assumptions of a constant value of μ and a single age at birth generated a birth stream that was very close to the birth stream in the full renewal equation. While still assuming a constant age at childbearing, the simulation depicted by the red line in Figure 4 takes the reproductive interval between 12 and 55 into account.

We assessed the accuracy of the approximation in Eq. (11) by simulation: We used the fertility and mortality age structure of women in Colombia from 1985 onward. The time axis was centered when NRR = 1. Thus, the year 2008 corresponds to t=0. The simulation started with a hypothetical NRR of 8 and a constant annual rate of decline (k) of 1.91%, while keeping the fertility age structure constant. The initial population age-structure was the stable-equivalent of the 1985 female Colombian population. The mean age at childbearing was 27 years. This is close to the average from our observation window of interest (1985-2008) of 26.9 years. Although we only fixed the level of the number of births to coincide with the actually observed number of female births in the year 2008 (t=0), the general shapes, levels, and modal ages in both curves match remarkably well. These findings also mirror Coale's (1972, p.121–122) earlier results. The deviations at the beginning can be attributed to the age structure: While the analytical trajectory assumes a pseudostable age structure, we used the stable equivalent age structure of the underlying fertility and mortality schedule, i.e., the right eigenvector to the dominant eigenvalue of the projection matrix.

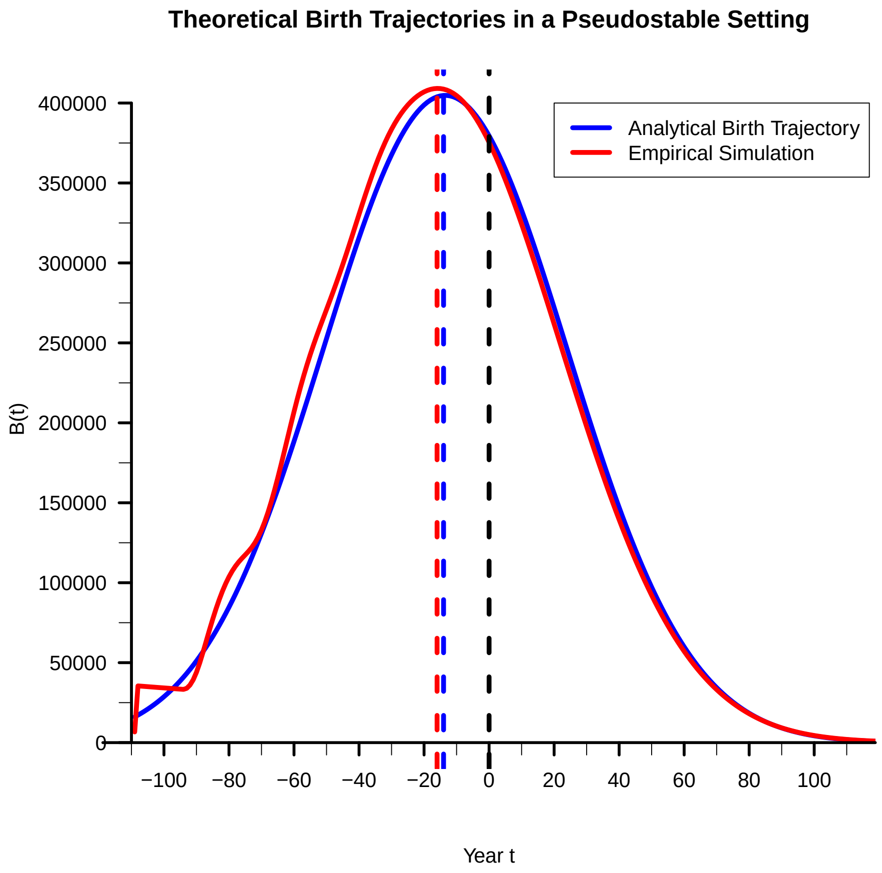

Figure 4: Births $B(t)$: A comparison of the analytical approximation (red) with an empirical simulation (blue).

In the remainder of this short introduction to pseudostable populations, we derive its time-dependent age structure $c(a,t)$ that results from the quadratic exponential birth sequence (11).

$$c(a,t) = \frac{N(a,t)}{N(t)} = \frac{B(t-a)l(a)}{\int_0^{\omega} B(t-x)l(x)\mathrm{d}x} = \frac{g(a,t)l(a)}{\int_0^{\omega} g(x,t)l(x)\mathrm{d}x}, \tag{12}$$

where $N(a,t)$ is the number of a years old females at time t, and

$$g(a,t) = \exp\left[-\frac{ak}{2} + \frac{a^2k}{2\mu} - \frac{kta}{\mu}\right]. \tag{13}$$

When fertility declines, there is a transitory period of low dependency. This occurs when the largest cohorts born during the transition are of prime working ages and there are few children and simultaneously few elderly. Fent et al. (2022) provide a formal framework for understanding the timing, duration, and magnitude of the support ratio that accompanies the decline in fertility. Since the classical stable population theory is not capable of carrying out such an analysis, the authors resorted to the pseudostable approach, which showed how the survival schedule, the pace of fertility decline and the generation length determine the timing, duration, and magnitude of the demographic support ratio. Note that this delivers a valuable contribution to the lasting discussion around the demographic dividend.

4 The Pseudostable Momentum

Inserting the pseudostable age structure (12) for the proportionate age distribution $c(a,t)$, we obtain from (6)

$$M(t) = \frac{e_0}{\mu} \frac{\int_0^{\mu} g(a,t)\mathrm{d}a}{\int_0^{\omega} g(a,t)l(a)\mathrm{d}a} \tag{14}$$

This section looks at the behavior of the momentum over the entire time interval from minus infinity to plus infinity. We start with an intuitive approach that delivers qualitative insights before proceeding in a stepwise manner to obtain quantitative results.

In the remote past, the high NRR led to a very young age distribution. Since there were scacely any older people living at that time, the ratio of integrals in (14) went to 1 for t versus minus infinity. O n t he o ther h a nd, t he a ge c omposition w as i ncreasingly c oncentrated to higher ages for very large periods t. Thus, the proportion of the pseudostable population in the age group from 0 to μ converged to zero.

This asymptotic behavior suggests a monotonous decrease of the momentum from e_0/μ to 0. In the following, we assume that $e_0/\mu > 1$ as life expectancy at birth is usually larger than the mean age at childbearing. Nevertheless, some exceptions have been observed pre-viously, for instance due to high infant mortality.[3]

Thus, interestingly enough, there should be a time $\hat{t}$ such that $M(\hat{t}) = 1$. Intuitively, this is a remarkable feature of pseudostable populations. Initially, i.e. in the remote past, there is a huge positive momentum. For replacement conditions imposed in the remote past, the pseudostable population may increase substantially (not to infinity as one could presume at the first glance, but) to a level of 2–3 times the initial population. In the next step, the positive momentum gradually declines until it reaches a time, denoted above as $\hat{t}$, where there is neither a positive nor a negative further growth of the population. Since $NRR(t) > 1$ for $t < 0$, the momentum at time 0 is still positive, i.e., $M(0) > 1$ (note that $NRR(0) = 1$).

From $\hat{t}$ onwards, the momentum is negative, which means that an already shrinking pop-ulation continues to shrink for a while until stationarity on a level smaller than 1 is even-tually reached. Note the asymmetry in the behavior of $M(t)$ for very large positive t as opposed to very large negative times. While in the first case the (negative) impact of de-clining fertility is restricted to 100%, the maximal (positive) momentum amounts to 200% to 300% in the latter.

Clearly, such considerations of plausibility do not replace a mathematical proof. From (14), we get

$$M(t) = \frac{e_0}{\mu} \frac{\mathcal{N}(t)}{\mathcal{D}(t)}. \tag{15}$$

where the numerator $\mathcal{N}(t)$ and the denominator $\mathcal{D}(t)$ of (15) are given by

$$\mathcal{N}(t) = \int_0^{\mu} g(a,t)\mathrm{d}a, \qquad \mathcal{D}(t) = \int_0^{\omega} g(a,t)l(a)\mathrm{d}a \tag{16}$$

[3]We checked the 2019 Revision of the World Population Prospects of the United Nations. Those were the ex-ceptions across all countries of the world during the years 1950 through 2020 (in alphabetical order): Afghanistan 1950-55; Cambodia 1975-80; Mali 1950-55, 1955-60; Rwanda 1990-95; Yemen 1950-55, 1955-60.

with

$$g(a,t) = \exp\left[k\left(-\tau a + \frac{a^2}{2\mu}\right)\right] \tag{17}$$

and $\tau = \frac{1}{2} + \frac{t}{\mu}$.

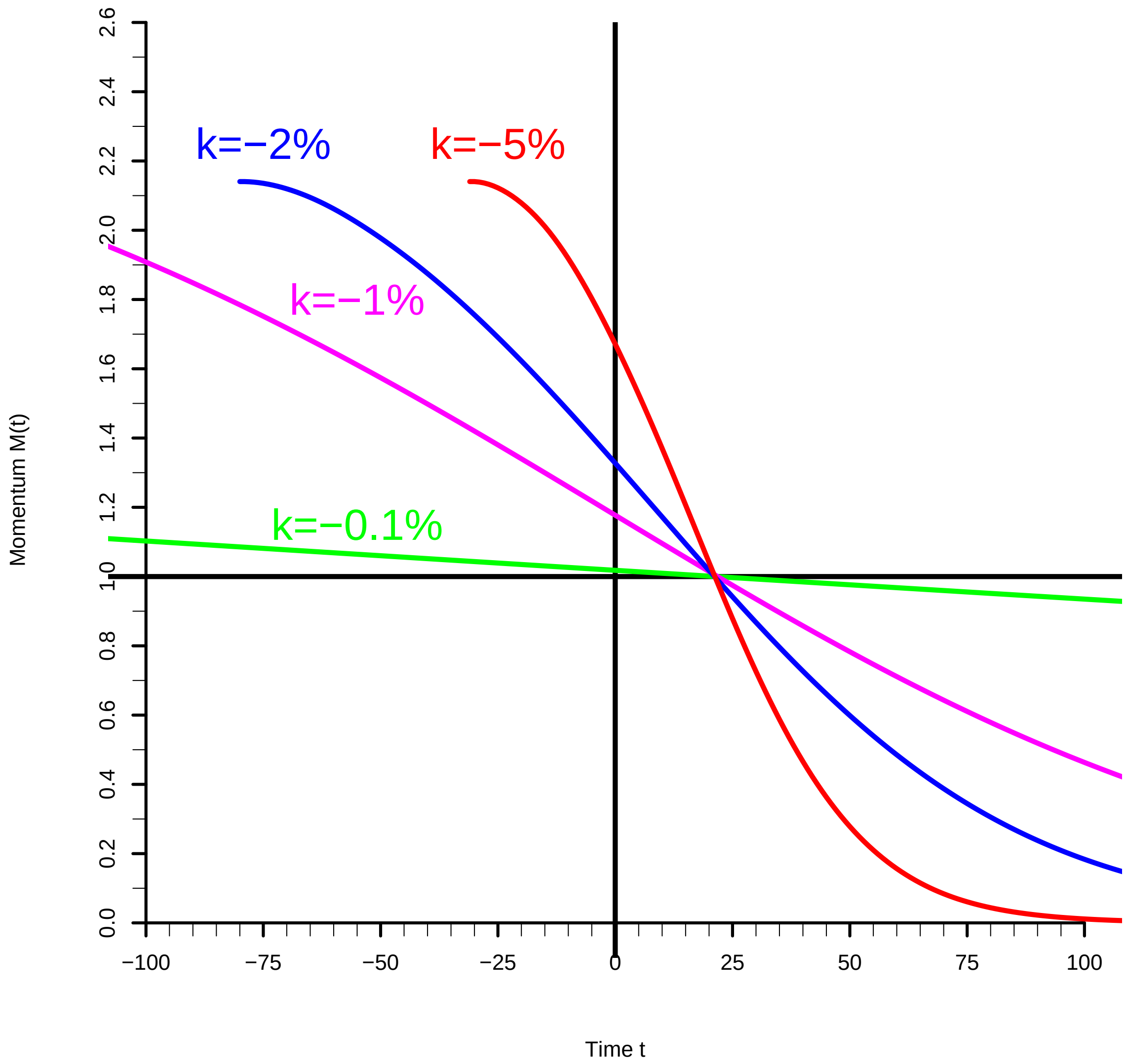

Figure 5: Time paths of the population momentum $M(t)$ in pseudostable populations with varying levels of fertility decline k.

With the timescale anchored ($t = 0$) when $NRR = 1$, Figure 5 illustrates the time path of the population momentum in pseudostable populations for four different values of the rate of fertility decline k. They range from an expedited reduction in fertility (-5% p.a.) to an extremely slow pace (-0.1% p.a.): For instance, it would only take 7.9 years to observe a decline in the NRR from 3 to 2 in the fast scenario, but more than 400 in the slow scenario. We would like to point out two interesting characteristics: 1) Despite this large variation in the pace of fertility decline, the "tipping point" for $M(t)$ is independent of k: They all converge at the same time-point. 2) In contrast to the canonical stable model, a $NRR = 1$ is not equivalent to $M = 1$ in the pseudostable case. Expressed differently: There is a time interval from $t = 0$ to $\tilde{t}$ where the observed NRR would induce a population decline in the stable model whereas it would trigger further population increase in the pseudostable model.

Figure 6 shows the long-run perspective of the population momentum. We simulated this long-term trajectory with the age-specific s urvival o f C olombian w omen i n 1 985 and their fertility age structure. The mean age at childbearing that year was 27.79 years, while life expectancy at birth was 71.12 years. As one probably expects, the momentum $M(t)$ tends to zero with ever-declining fertility. To our initial surprise, however, momentum in the remote past does not increase infinitely w ith c ontinously h igher f ertility. Instead, it bends over to an asymptotic limit of e_0/μ (see also Proposition 1). With a mean age at childbearing of 27.79 years and life expectancy at birth of 71.12, the *theoretical* upper value of $M(t \to -\infty) = e_0/\mu = 71.12/27.79 = 2.56$. Our *empirical* estimate (with $k = -0.5\%$) led to a $M(t \to -\infty) = 2.55$.

Next, we are interested in how the pseudostable momentum depends on time t. By differ-entiating equation (15) with respect to t and applying the quotient rule, we obtain

$$M'(t) = M(t) \left(\frac{\mathcal{N}'(t)}{\mathcal{N}(t)} - \frac{\mathcal{D}'(t)}{\mathcal{D}(t)} \right) \tag{18}$$

Direct calculations lead to

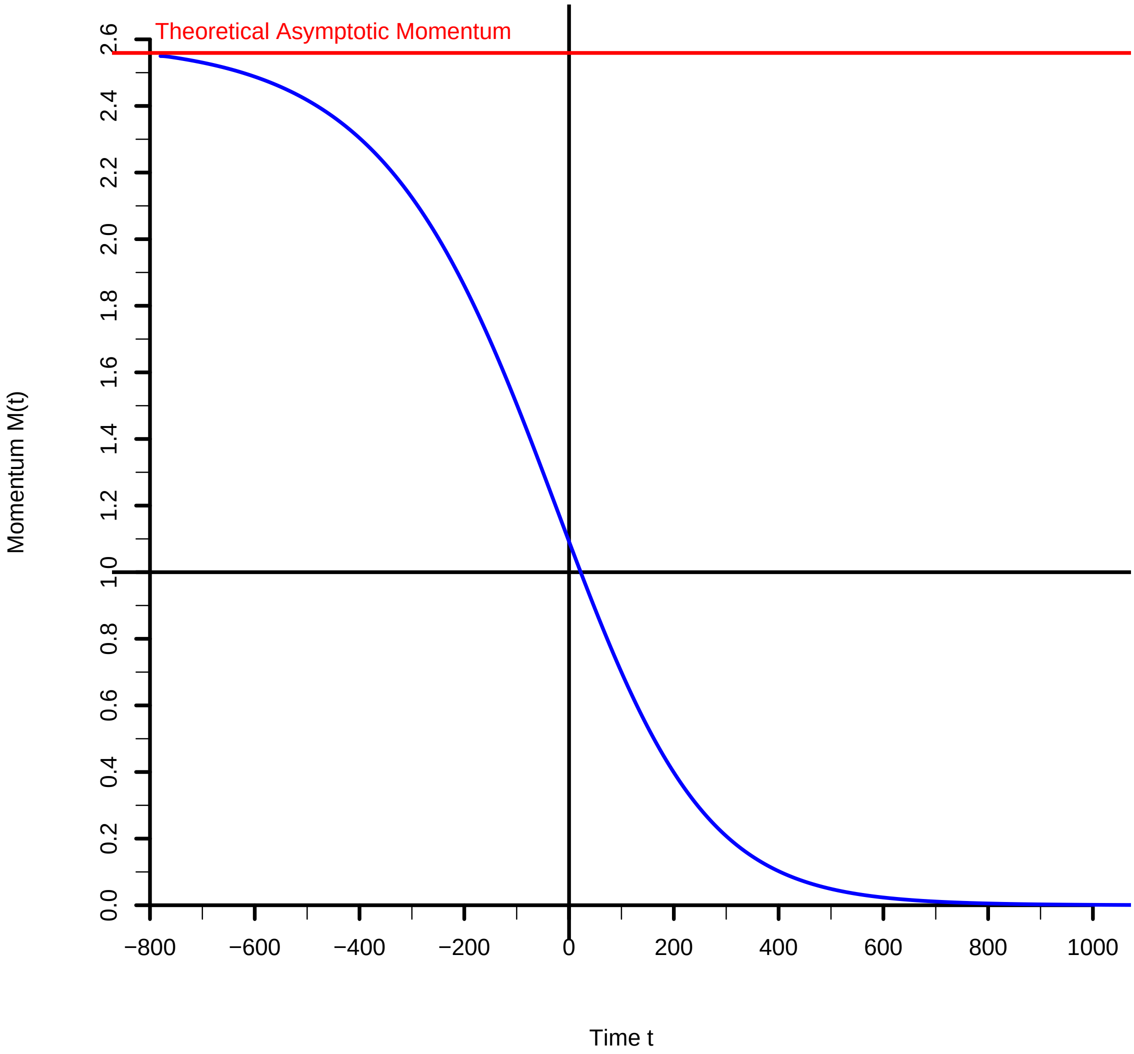

Figure 6: Population momentum $M(t)$ — long term perspective.

$$\mathcal{N}'(t) = \int_0^{\mu} g(a,t) \left(\frac{-ka}{\mu} \right) \mathrm{d}a = -\frac{k}{\mu} \int_0^{\mu} a g(a,t) \mathrm{d}a \tag{19}$$

and

$$\mathcal{D}'(t) = \int_0^{\omega} g(a,t) \left(\frac{-ka}{\mu} \right) l(a) \mathrm{d}a = -\frac{k}{\mu} \int_0^{\omega} a g(a,t) l(a) \mathrm{d}a \tag{20}$$

Therefore (18) can be rewritten as

$$M'(t) = \left(\frac{k}{\mu}\right) M(t) \left(\frac{\int_0^{\omega} ag(a,t)l(a)\mathrm{d}a}{\int_0^{\omega} g(a,t)l(a)\mathrm{d}a} - \frac{\int_0^{\mu} ag(a,t)\mathrm{d}a}{\int_0^{\mu} g(a,t)\mathrm{d}a} \right) = \left(\frac{k}{\mu}\right) M(t) \left(A(t) - A_{\mu}^{0}(t) \right), \tag{21}$$

with

$$A(t) = \frac{\int_0^{\omega} ag(a,t)l(a)\mathrm{d}a}{\int_0^{\omega} g(a,t)l(a)\mathrm{d}a} \quad \text{and} \quad A_{\mu}^{0}(t) = \frac{\int_0^{\mu} ag(a,t)\mathrm{d}a}{\int_0^{\mu} g(a,t)\mathrm{d}a} \tag{22}$$

where $A(t)$ is the mean age of the total pseudostable population under a general survival function and $A_{\mu}^{0}(t)$ denotes the mean age of those females below age μ under a rectangular survival function.

We conjecture that $A(t) > A_{\mu}^{0}(t)$ and together with $k < 0$, we conclude that $M'(t) < 0$. This is also supported by numerical simulations.

The following proposition collects the structural properties of the momentum for pseudostable populations. It constitutes the main results of the paper.

Proposition 1: *The pseudostable momentum $M(t)$ develops as follows:*

(a) $\lim_{t \to -\infty} M(t) = e_0/\mu$,

(b) $\lim_{t \to +\infty} M(t) = 0$,

(c) $dM(t)/dt < 0$;

(d) there exists a unique time $\hat{t}$ such that $M(\hat{t}) = 1$;

(e) there is a unique point of inflection $\tilde{t}$ characterized by $M''(\tilde{t}) = 0$.[4]

Following equation (6) and by taking $\int_0^{\omega} c(t,a)\mathrm{d}a = 1$ into consideration, rectangular mortality, and $\mu < \omega = e_0$ we conclude that $M(t) < e_0/\mu$. An intuitive explanation of the asymptotic result (a) from Proposition 1 by using the renewal theorem is the following[5]: One new born girl $B = 1$ leads to a birth rate $1/\mu$. Thus, the ultimate population equals $Be_0/\mu = e_0/\mu$.

[4]The uniqueness is only supported by numerical simulations.

[5]We are grateful to Joshua R. Goldstein, UC Berkeley, for suggesting this interpretation.

Figure 7 shows the pseudo-stable population momentum $M(t)$ in Panel A) and its first and second derivatives in panels B) and C), respectively. Plots are based on analytic expressions. The following assumptions were made: Mean age at childbearing in a stationary population $\mu = 30$; proportional fertility reduction per unit of time $k = -0.025$; Gompertz-mortality with parameters $\alpha = 10^{-5}$, and $\beta = 0.11$, resulting in life expectancy at birth e_0=79.33.

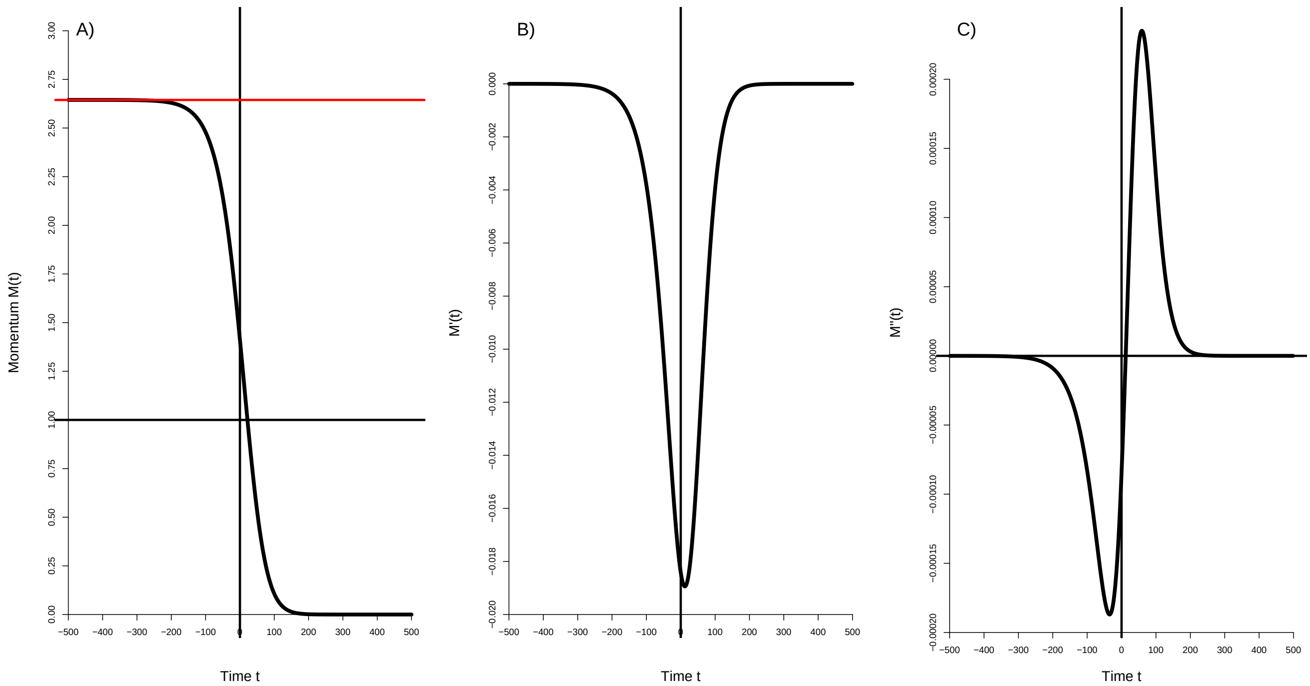

Figure 7: Momentum and its derivatives depending on time.

Remark: For a rectangular survival function $l(a)$, i.e.

$$l(a) = \begin{cases} 1 & \text{if} \quad a < \omega \\ 0 & \text{if} \quad \omega \le a \end{cases} \tag{23}$$

the mean age of the total pseudostable population reduces to

$$A^0(t) = \frac{\int_0^\omega a g(a,t) \mathrm{d}a}{\int_0^\omega g(a,t) \mathrm{d}a} \tag{24}$$

Clearly, it holds that $A^0(t) > A^0_\mu(t)$,which proves point (c) of proposition 1 for rectangu-lar survival functions. [6]

Next, we will sketch the significance of the results from proposition 1. Point (a) describes the remarkable fact, that the asymptotic behavior of the pseudostable momentum $M(t)$ in the remote past does not depend on the shrinking rate k of reproduction. Moreover, the limit is given by the quotient of the upper age limit ω and the mean age of birth μ. According to (c) and (b) $M(t)$ decreases montonously before converging finally to zero.

While the momentum could initially be presumed as infinitely large in the very remote past, this is not true. In our model, however, the momentum decreases gradually from its maximal value e_0/μ to zero. Point (d) says that there is a time $\hat{t}$ in between, where there no momentum exists. For $t < \hat{t}$, there is a 'positive' momentum (or, more formally: the momentum exceeds one), whereas the momentum is 'negative' for $t > \hat{t}$ i.e., the population continues to shrink for a while if the net reproduction rate immediately increases to replacement level. Since the net reproduction rate $NRR(0)$ equals 1 and was higher in the past, there are enough potential mothers to imply a positive momentum for $t = 0$, i. e, $M(0) > 1$.

For the proof of proposition 1, we complete the expression in the exponential function (13) of the exponent in $g(a, t)$ to a square by adding $(\mu/2)\tau^2$. To establish the first two points (a) and (b), the rule of de L'Hôspital is applied. The proof of (c) uses the generalized mean value theorem of differential calculus. For details also of the additional points raised in the proposition see Appendix B. Together with the asymptotic behavior, this establishes the existence of at least one point of inflection. To ascertain it, we set $M''(t) = 0$.

Next, we vary the core parameter of the pseudostable model, the rate of fertility decline k. The results are summarized in the following

Proposition 2: *Consider the momentum not only as time function but also depending on the rate k of declining fertility, i.e.,* $M(t; k)$.

For decreasing absolute values of the parameter k, the momentum $M(t; k)$ *becomes flatter, while*

[6] For an analytical proof of Proposition 1 in case of rectangular survival functions, see Appendix B.

larger declining rates lead to a steeper momentum.

In the limit one obtains a horizontal straight line and a step function, respectively.

For $k \to -\infty$, *it holds that*

$$M(t;k) = \begin{cases} e_0/\mu & \textit{for} \quad t < \mu/2 \\ 0 & \textit{for} \quad t > \mu/2 \end{cases} \tag{25}$$

while for $k \to 0$, *we have*

$$M(t;k) = 1 \textit{ for all } t. \tag{26}$$

The neutral time $\hat{t}$ *is a function of* k, *i.e.,* $\hat{t}(k)$, *which increases with* k, *i.e.,* $d\hat{t}(k)/dt > 0$ *(note that* $k < 0$*). Moreover,*

$$\lim_{k \to 0} \hat{t}(k) = +\infty, \qquad \lim_{k \to -\infty} \hat{t}(k) = \mu/2. \tag{27}$$

The proof is sketched in Appendix B.

Figure 8 depicts the momentum $M(t)$ for extremely high values of fertility decline k. The points of intersection with $M(t) = 1$ lead to the neutral times $\hat{t}(k)$. This figure also illustrates how the momentum gets steeper for $k \to -\infty$ and that $\hat{t}(k) \to \mu/2$. From simulations, we derive that the same asymptotic behavior is true for the point of inflection $\tilde{t} = \tilde{t}(k)$

Table 1 shows estimated values for the point of inflection $\tilde{t}$ as well as $\hat{t}$, i.e., the time point when momentum $M(t) = 1$, for selected levels of fertility reduction k. The results are based on the assumption of a mean age at childbearing $\mu = 30$ in stationary populations and mortality following a Gompertz distribution with parameters $\alpha = 0.00001$ and $\beta = 0.11$ resulting in life expectancy at birth $e_0 = 79.3$ years. Age 109 was used as the upper age-limit ω. This is the age at which there is at most one in a million survivors with the given parameters of the Gompertz distribution. Note that for k very near to zero (larger than -0.0020) the dependence both of $\hat{t}$ and $\tilde{t}$ on k is not monotonous, which seems to be due to numerical artefacts.

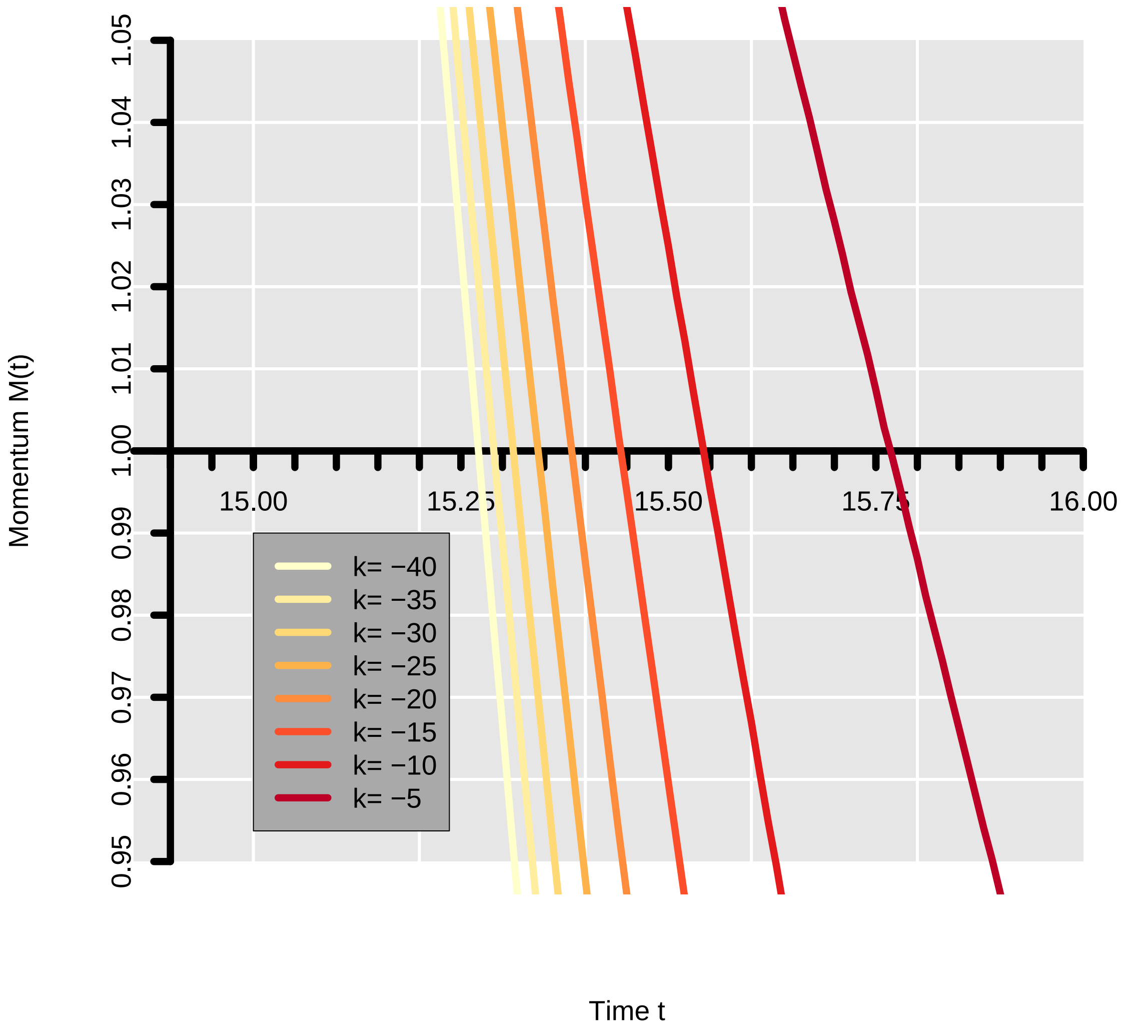

Figure 8: The momentum of an extremely high decline of fertility (i.e., extremely low values of k)

The time from $t = 0$ to $\hat{t}$ is interesting for several reasons. First, for $0 < t \leq \hat{t}$, fertility is below the replacement level. Nevertheless we observe *positive* momentum — that is counterintuitive to the standard case of population momentum. Second, $M(\hat{t}) = 1$ implies that the population size observed at $\hat{t}$ is equivalent to the population size obtained with an

Table 1: Neutral times $\hat{t}$ and time points of inflection $\tilde{t}$ of the momentum depending on the fertility decline, as measured by parameter k.

k	$\widehat{t}$	$\widetilde{t}$
-0.0001	18.060	22.076
-0.0002	20.400	21.804
-0.0005	21.790	22.018
-0.0010	22.240	22.439
-0.0020	22.440	22.889
-0.0050	22.480	22.503
-0.0100	22.360	22.378
-0.0200	22.070	22.044
-0.0500	21.230	21.252
-0.1000	20.150	20.158
-0.2000	18.850	18.872

Note the surprisingly small differences between the neutral time $\widehat{t}$ and the inflection time $\widetilde{t}$.

immediate switch to replacement level fertility at $\hat{t}$. Third, the age structure of the pseudo-stable population is *not* equivalent to the stationary age structure, as shown in Figure 9.

The stationary age structure features more children and more people at higher ages whereas the pseudo-stable population contains more persons at working ages. Expressed differently: All standard dependency ratios (young, old, total) are more favorable in the pseudo-stable population at $\hat{t}$ than in the stationary case.

5 Conclusions

Although stable populations are at the core of formal demography, they do not — to the best of our knowledge — occur in reality, where age-specific vitality rates change over time. Coale (1972) extended the stable framework by assuming uniformly declining fertility rates with-out giving up the flair of analytic tractability. In the current paper, we studied one special aspect of this type of populations, later denoted as pseudostable ones (see also Feichtinger and Vogelsang, 1978), namely their demographic momentum.

We derived a qualitative characterization of the time path of the momentum $M(t)$ of

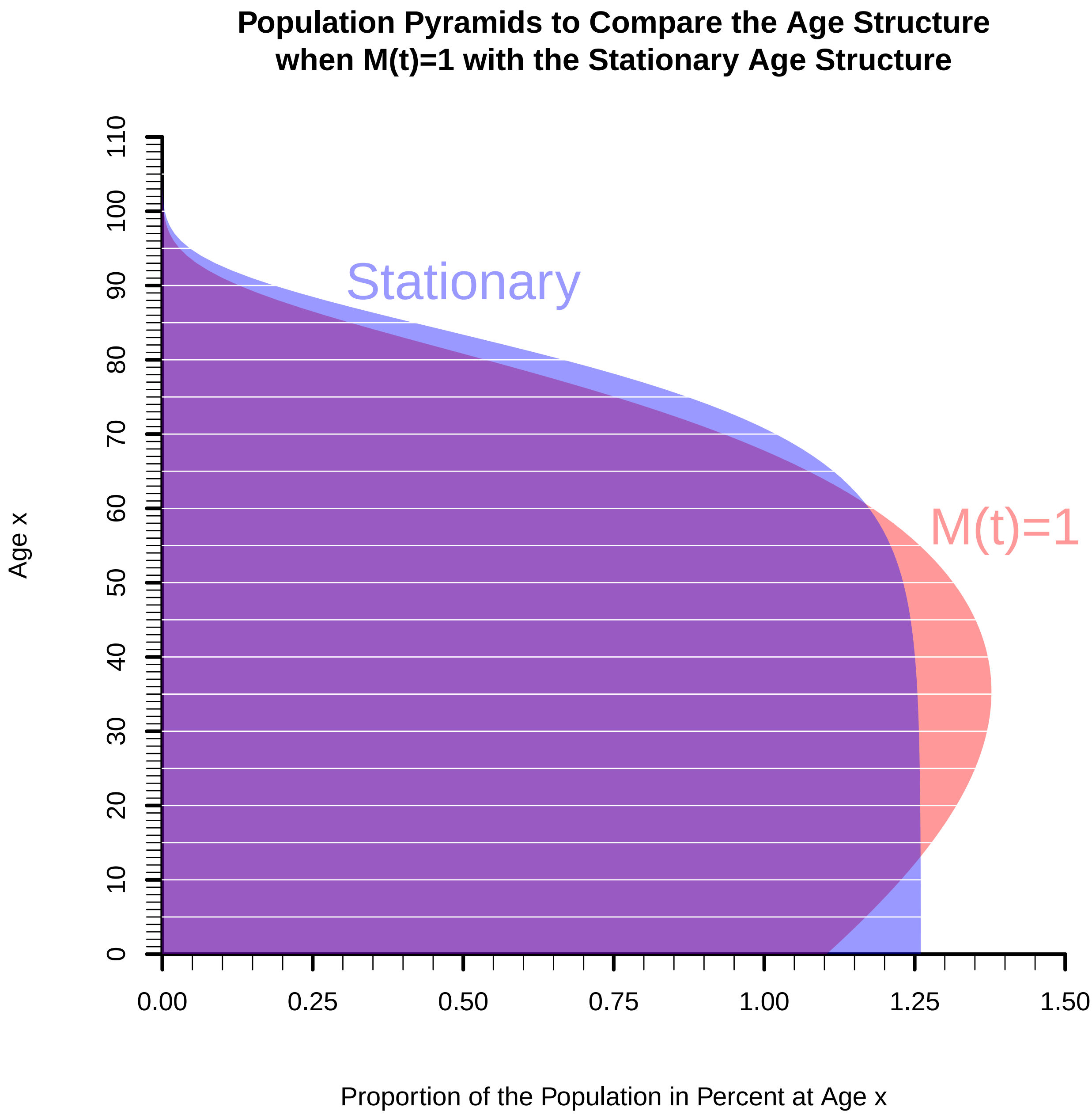

Figure 9: Population pyramide to compare the age structure when $M(t) = 1$ with the stationary age structure

population growth and decline under pseudostable conditions. It turned out that $M(t)$ is a monotonously decreasing $S-$shaped function that converges to zero for increasing times, but remarkably to a finite value given by the ratio e_0/μ in the remote past. This guarantees the existence of a time $\hat{t}$ with a neutral momentum, i.e. $M(\hat{t}) = 1$. To prove this interesting behavior of the pseudostable momentum, we not only had to assume a fixed unique birth age, but also a rectangular mortality schedule.

Moreover, we provided a dynamic sensitivity analysis with respect to the rate k of fertility decline. For decreasing absolute values of the parameter k, the momentum becomes flatter, while larger declining rates lead to a steeper momentum. In the limit, this results in a horizontal straight line and a step function jumping from e_0/μ to zero for $t = \mu/2$, respectively.

Reflecting the history of the natality and the mortality of a population, its age composition can be seen as its memory. The reason for the demographic momentum is an inertia of age structures containing a relatively large number of potential parents due to past high fertility. Although a miraculous, immediate reduction of fertility to replacement level is un-realistic, a gradual decline of fertility in fast growing populations seems inevitable. Since any delay in fertility decline to a stationary level leads to an increase of the momentum, it makes sense to think about the timing and the quantum of the reduction in reproduc-tion. More specifically, there is an intertemporal trade-off between costly family planning measures and the size of the demographic momentum at the end of the planning period. In Feichtinger and Wrzaczek (2022) a distributed parameter control framework is used to study this problem. An appropriate extension of Pontrayagin's maximum principle allows for deriving interesting insights into the qualitative structure of the optimal path of fertility control and the resulting salvage momentum. In particular, this approach can be applied in a symmetric way to determine efficient pro-natalistic measures for shrinking populations, a situation that is currently prevailing in several developed countries.

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Appendix A: Addendum to Pseudostable Populations

This Appendix section provides details on pseudostable populations. For more detailed informations on these interesting populations see Coale (1972, chap. 4) and Feichtinger and Vogelsang (1978).

We start with the derivation of the potentially surprising fact of a 'birth mountain' resulting from a permanent decline of fertility at a constant rate k.

Let us start with equation (10). For a constant $\mu(t) = \mu$ it may be written as

$$B(t) = R(0)\exp(kt)B(t-\mu). \tag{28}$$

Setting $Y(t) = \log B(t)$ and reiterating the assumption $R(0) = 1$, equation (28) may be written as

$$Y(t) - Y(t-\mu) = kt. \tag{29}$$

As mentioned in Section 3, this is a linear difference equation that may be solved by assuming $Y(t) = b_1 t + b_2 t^2$. Substituting this quadratic polynomial in (29) and solving for the coefficients b_1 and b_2, yields $b_1 = k/2$, $b_2 = k/(2\mu)$ and hence the birth path with the quadratic exponential previously described in (11).

Next, the mean age of the total population $N(t)$ is given by

$$A(t) = \int_0^\omega ac(a,t)\mathrm{d}a. \tag{30}$$

Combined with (12) and (13), this results in a permanent increase in the mean age of pseudostable populations, since it holds that

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = -\frac{k\sigma^2}{\mu}, \tag{31}$$

is greater than zero, where

$$\sigma^2(t) = \int_0^{\omega} (a - A(t))^2 c(a,t) \mathrm{d}a \tag{32}$$

is the variance of the age structure. Note that (31) results from the linear approximation of the exponential function (13) (see Feichtinger and Vogelsang (1978, p.39)) for the stable pendant. (compare also Keyfitz, 1977, p.88–89).

Note that the following limits are given by

$$\lim_{t \to +\infty} A(t) = \omega, \qquad \lim_{t \to -\infty} A(t) = 0.$$

An efficient tool created by Coale (1972) to analyze populations with a constantly changing net reproduction rate (i.e. pseudostable populations) is to compare them at an arbitrary time t with the stable population resulting from the fertility schedule prevailing in t. The growth rate of the stable equivalent is given as a unique real solution to the characteristic equation,

$$\int_0^{\omega} \exp(-r_s(t)) l(a) m(a,t) \mathrm{d}a = 1,$$

namely

$$r_s(t) = \frac{\log R(t)}{\mu} = \frac{kt}{\mu}. \tag{33}$$

Appendix B: Proofs

Under the restriction of a rectangular survival function $l(a)$ (see eq. (23)), we provide a formal proof.

Proof of Proposition 1:

By completing the expression in the exp-function (13) to a square the pseudostable momentum given in (14) is proportional to

$$\frac{\int_0^\mu g(a,t)\mathrm{d}a}{\int_0^\omega g(a,t)\mathrm{d}a} = \frac{\exp(k\mu\tau^2/2)\int_0^\mu \exp\left[k\left(-\tau a + \frac{a^2}{2\mu}\right)\right]\mathrm{d}a}{\exp(k\mu\tau^2/2)\int_0^\omega \exp\left[k\left(-\tau a + \frac{a^2}{2\mu}\right)\right]\mathrm{d}a} = \frac{\int_0^\mu \exp\left[\frac{k(a-\mu\tau)^2}{2\mu}\right]\mathrm{d}a}{\int_0^\omega \exp\left[\frac{k(a-\mu\tau)^2}{2\mu}\right]\mathrm{d}a} \tag{34}$$

Using this in (15) yields a quotient of two well-known integrals of the form $\int \exp(-z^2)\mathrm{d}z$. As

$$\int \exp\left[\frac{k(a-\mu\tau)^2}{2\mu}\right]\mathrm{d}a = \int \exp\left[-\left(\sqrt{\frac{-k}{2\mu}}(a-\mu\tau)\right)^2\right]\mathrm{d}a \tag{35}$$

we therefore transform a to z by

$$z = \sqrt{\left(\frac{-k}{2\mu}\right)}(a-\tau\mu) \tag{36}$$

and obtain a simpler integrand, but more complicated integration limits. This transformation also leads to

$$\mathrm{d}a = \mathrm{d}z\sqrt{\frac{2\mu}{-k}} \tag{37}$$

Thus the integral in (1) is equivalent to

$$\int \exp\left[-\left(\sqrt{\frac{-k}{2\mu}}(a-\mu\tau)\right)^2\right]\mathrm{d}a = \sqrt{\frac{2\mu}{-k}}\int \exp\left[-z^2\right]\mathrm{d}z \tag{38}$$

The integration limits have to be changed in the following way:

$$a = 0 \Rightarrow z = -\tau\sqrt{\frac{-k\mu}{2}}, \quad a = \mu \Rightarrow z = (1-\tau)\sqrt{\frac{-k\mu}{2}}, \quad a = \omega \Rightarrow z = \left(\frac{\omega}{\mu} - \tau\right)\sqrt{\frac{-k\mu}{2}} \tag{39}$$

Therefore, we next use the momentum $M(t)$ transformed as indicated, ignoring the factor ω/μ, and denote it as

$$J(x;u,v) = \frac{\int_x^{x+u}\exp[-z^2]\mathrm{d}z}{\int_x^{x+v}\exp[-z^2]\mathrm{d}z}, \qquad 0 < u < v. \tag{40}$$

with

$$x = -\tau\sqrt{\frac{-k\mu}{2}}, \qquad u = \sqrt{\frac{-k\mu}{2}}, \qquad v = \frac{\omega}{\mu}\sqrt{\frac{-k\mu}{2}} \tag{41}$$

Now, we aim to show that

$$\frac{\partial J(x;u,v)}{\partial x} > 0. \tag{42}$$

Note that the sign of the derivative in (42) is opposite to that in point (c) of Proposition 1. This is because t (τ) and z are related by (36).

To establish (42), we consider the following function

$$F(x,y) = -\log\left[\exp(x^2)\int_x^{x+y}\exp(-z^2)\mathrm{d}z\right], \quad y > 0 \tag{43}$$

Direct calculation shows the validity of

$$\frac{\partial \log J(x;u,v)}{\partial x} = F_x(x,v) - F_x(x,u). \tag{44}$$

This follows from

$$J(x,u,v) = \frac{\exp(x^2)\int_x^{x+u}\exp[-z^2]\mathrm{d}z}{\exp(x^2)\int_x^{x+v}\exp[-z^2]\mathrm{d}z}, \qquad \Rightarrow \log J(x,u,v) = F(x,u) - F(x,v) \tag{45}$$

where

$$F_x(x,y) = -2x + \frac{\exp(-x^2) - \exp(-(x+y)^2)}{\int_x^{x+y}\exp(-z^2)\mathrm{d}z} \tag{46}$$

Now we prove two lemmata for the function (43).

Lemma A.1: There exists a $\delta \in (0,y)$ such that

$$F_x(x,y) = 2\delta. \tag{47}$$

Proof:

$$\frac{\exp(-(x+y)^2)-\exp(-x^2)}{\int_x^{x+y}\exp(-z^2)\mathrm{d}z} = \frac{\exp(-(x+y)^2)-\exp(-x^2)}{\int_0^{x+y}\exp(-z^2)\mathrm{d}z - \int_0^{x}\exp(-z^2)\mathrm{d}z} \tag{48}$$

Defining the function $f(x) = \exp(-x^2)$ and $g(x) = \int_0^x \exp(-z^2)\mathrm{d}z$ then according to the generalized mean value theorem of calculus[7] there exists a $\delta \in (0, y)$ such that

$$\frac{\exp(-(x+y)^2)-\exp(-x^2)}{\int_0^{x+y}\exp(-z^2)\mathrm{d}z - \int_0^{x}\exp(-z^2)\mathrm{d}z} = \frac{-2(x+\delta)\exp[(-x+\delta)^2]}{\exp[-(x+\delta)^2]} = -2(x+\delta) \tag{50}$$

leading to $F_x(x,y) = -2x + 2(x+\delta) = 2\delta$.

Lemma A.2: For $0 < u < v$ it holds that

$$F_x(x,u) < F_x(x,v), \quad \forall x \in \mathbb{R} \tag{51}$$

which means that $F_x(x,y)$ is monotonically increasing in y for fixed x.

Proof:

$$F_{xy} = \frac{2(x+y)\exp[-(x+y)^2]}{\int_x^{x+y}\exp[-z^2]\mathrm{d}z} - \frac{\left(\exp[-x^2]-\exp[-(x+y)^2]\right)\exp[-(x+y)^2]}{\left(\int_x^{x+y}\exp[-z^2]\mathrm{d}z\right)^2} \tag{52}$$

Simplifying and applying the above result leads to

$$F_{xy} = \frac{\exp[-(x+y)^2]}{\int_x^{x+y}\exp[-z^2]\mathrm{d}z}\left[2(x+y)-(F_x+2x)\right] = \frac{2(y-\delta)\exp[-(x+y)^2]}{\int_x^{x+y}\exp[-z^2]\mathrm{d}z} > 0 \tag{53}$$

(51) and (44) deliver the desired result (42). Hence, this establishes the monotonic decrease of $M(t)$ asserted in (c) of Proposition 1.

To prove the asymptotic behavior declared in points (a) and (b) the rule of de L'Hôspital

[7]For sufficiently smooth functions $f(x)$ and $g(x)$, there exists a value $x_0 \in (\alpha, \beta)$ such that

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(x_0)}{g'(x_0)}. \tag{49}$$

can be applied.

Remark: Remember that two approximations, namely (10) with $\mu(t) = \mu$ and (23) , led us to the nice structural properties of the pseudostable momentum and the related times. Numerical simulations are required to establish the validity of the results from Proposition 1 in a more general setting, i. e., for realistic (non-rectangular) mortality schedules.

To sketch the proof of proposition 2, we rewrite $M(t)$ as

$$J(p, p(x+c)) = \frac{\int_{px}^{p(x+a)} \exp(-z^2)\mathrm{d}z}{\int_{px}^{p(x+b)} \exp(-z^2)\mathrm{d}z} \tag{54}$$

with $p = -k/2\mu$, and $c = a$ or b with $a < b$. Note that $x = -(\mu/2 + t)$.

Now, let $p \to \infty$. Then, we have to distinguish between the following four cases:

(a) $0 < px < p(x+c)$

(b) $px < p(x+c) < 0$

(c) $px < p(x+a) < 0 < p(x+b)$

(d) $px < 0 < p(x+a) < p(x+b)$.

The first two cases result in a limes of '0/0' and can be solved by using the rule of de L'Hôspital appropriately.

The remaining two cases use the well-known fact that

$$\int_{-\infty}^{+\infty} \exp(-z^2)dz = \sqrt{\pi}. \tag{55}$$

Remarkably, for $p \to \infty$ (18) has a jump for $x = -\mu$, which means that the momentum $M(t)$ jumps from ω/μ to 0 at the time $t = \mu/2$.

Appendix C: Additional Material

Differentiating relation (18) with respect to time and taking into consideration the derivatives of the mean ages

$$A'(t) = -\frac{k\sigma^2(t)}{\mu}, \qquad A'_\mu(t) = -\frac{k\sigma^2_\mu(t)}{\mu}$$

with the pertinent variances yields

$$M''(t) = \frac{k}{\mu}\left\{M'(t)[A(t) - A_\mu(t)] - \frac{k}{\mu}M(t)[\sigma^2(t) - \sigma^2_\mu(t)]\right\} \tag{56}$$

Setting (56) equal to zero, results in the equation

$$\frac{M'(t)}{M(t)} = \frac{k}{\mu}\frac{[\sigma^2(t) - \sigma^2_\mu(t)]}{[A(t) - A_\mu(t)]} \tag{57}$$

to determine the time $\tilde{t}$ where $M''(\tilde{t}) = 0$, i.e., the point of inflection occurs.