Electronic Transactions on Numerical Analysis.
Volume 58, pp. 244–270, 2023.

ISSN 1068–9613.
DOI: 10.1553/etna_vol58s244

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Kent State University and
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ON CVETKOVIĆ-KOSTIĆ-VARGA TYPE MATRICES*

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Abstract. Cvetković-Kostić-Varga (CKV)-type matrices play a significant role in numerical linear algebra. However, verifying whether a given matrix is a CKV-type matrix is complicated because it involves choosing a suitable subset of $\{1, 2, \ldots, n\}$. In this paper, we give some easily computable and verifiable equivalent conditions for a CKV-type matrix, and based on these conditions, two direct algorithms with less computational cost for identifying CKV-type matrices are put forward. Moreover, by considering the matrix sparsity pattern, two classes of matrices called S-Sparse Ostrowski-Brauer type-I and type-II matrices are proposed and then proved to be subclasses of CKV-type matrices. The relationships with other subclasses of H-matrices are also discussed. Besides, a new eigenvalue localization set involving the sparsity pattern for matrices is presented, which requires less computational cost than that provided by Cvetković et al. [Linear Algebra Appl., 608 (2021), pp.158–184].

Key words. CKV-type matrices, S-Sparse Ostrowski-Brauer type-I matrices, S-Sparse Ostrowski-Brauer type-II matrices, H-matrices

AMS subject classifications. 15A18, 15A42, 15A69

1. Introduction. Let $\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ be the set of all $n \times n$ complex (real) matrices, $N := \{1, \ldots, n\}$, and let $|N|$ be the cardinality of N. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a nonsingular H-matrix [2] if its comparison matrix $\langle A\rangle = [m_{ij}] \in \mathbb{R}^{n\times n}$ defined by

$$m_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j \end{cases}$$

is a nonsingular M-matrix, i.e., $\langle A\rangle^{-1} \geq 0$. H-matrices are widely used in many areas such as computational mathematics, economics, mathematical physics, and dynamical system theory [2, 7, 30]. An interesting topic, among others, is to explore the subclasses of H-matrices, as several applied linear algebra research areas such as the Schur complement problem [10, 16, 21, 24, 31], the subdirect sum problem [3, 4, 14], and the estimation of error bounds for linear complementarity problems [5, 13, 15, 28], etc., are closely connected with special subclasses of nonsingular H-matrices.

Recall an important class of matrices: S-strictly diagonally dominant matrices, that is, for a given nonempty subset S of N, a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an S-strictly diagonally dominant (S-SDD) matrix [9] if for all $i \in S$ and $j \in \overline{S}$,

$$|a_{ii}| > r_i^S(A) \quad \text{and} \quad \big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > r_i^{\overline{S}}(A) r_j^S(A),$$

where $r_i^S(A) := \sum\limits_{k\in S\setminus\{i\}} |a_{ik}|$ and $\overline{S} := N \setminus S$. In [9], Cvetković, Kostić, and Varga proved that S-SDD matrices are nonsingular H-matrices and applied this property to give a new eigenvalue localization set for matrices in the complex plane. Moreover, as in [7] and [8], the union of all S-SDD matrices is usually called Cvetković-Kostić-Varga (CKV) class or Σ-SDD class, that is, a matrix A belongs to the class of CKV matrices if there exists a nonempty proper subset S of N such that A is an S-SDD matrix.

*Received November 22, 2021, Accepted January 18, 2023. Published online on March 6, 2023. Recommended by Froilan Dopico.
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Very recently, Cvetković et al. discovered a new subclass of nonsingular H-matrices called CKV-type matrices, which generalizes the CKV class.

DEFINITION 1.1 ([8]). *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a CKV-type matrix if $N_- = \emptyset$ or $S_i^\star(A)$ is not empty for all $i \in N_-$, where $N_- := \{i \in N : |a_{ii}| \le r_i(A)\}$ and*

$$S_i^\star(A) := \Big\{ S \in \Sigma(i) : |a_{ii}| > r_i^S(A), \text{ and for all } j \in \overline{S} \quad \big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > r_i^{\overline{S}}(A) r_j^S(A) \Big\},$$

with $\Sigma(i) := \{S \subsetneq N : i \in S\}$.

As reported in [8], the CKV-type class has potential applications in many fields of numerical linear algebra such as eigenvalue localization, infinity-norm bound for the inverse matrix, and pseudospectra localization, among many other problems. It is well-known (see [1, 17, 29]) that for many applications it is useful to know whether a given matrix is an H-matrix, and up to now, many direct and iterative algorithms such as Noda iterations [17], Algorithm AH [1], and Algorithm YZ [29] have been developed for determining the H-matrix characterization of the coefficient matrix in a linear system or a linear complementarity problem (LCP). However, in some cases it is not enough to know this, but it is also necessary to know whether a given matrix is in a special subclass of the H-matrices, e.g., in a CKV-type class. For instance, by the Schur complement of matrices, the problem of solving a large-scale linear system

$$A\mathbf{x} = \mathbf{b}$$

with compatible partitioning

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\top, \qquad \mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)^\top$$

often transforms into solving the following two smaller linear systems

$$A_{11}\mathbf{x}_1 = \mathbf{b}_1 - A_{12}\mathbf{x}_2, \tag{1.1}$$

$$A/A_{11}\mathbf{x}_2 = \mathbf{b}_2 - A_{21}A_{11}^{-1}\mathbf{b}_1, \tag{1.2}$$

where A_{11} is nonsingular and $A/A_{11} := (A_{22} - A_{21}A_{11}^{-1}A_{12})$ is the Schur complement of A concerning A_{11}. A method used to solve (1.1) and (1.2) instead of directly solving the original equation $A\mathbf{x} = \mathbf{b}$ is usually called the Schur-based method [21, 22, 23]. When the coefficient matrix of the linear system belongs to a special subclass of H-matrices, the Schur-based method might possess nice convergence properties, and in some cases it is more effective than directly using classical methods such as the Gauss-Seidel (GS) method and the conjugate gradient (CG) method, etc. On the other hand, it is known that special subclasses of H-matrices with positive diagonal entries play a particularly important role in estimating the error for the solution of the LCP [6, 13, 15, 28].

Although Definition 1.1 provides a criterion for identifying CKV-type matrices by checking whether $N_- = \emptyset$ or $S_i^\star(A)$ (for all $i \in N_-$) is not empty over all possible $S \in \Sigma(i)$, it may not be suitable for large matrices because the cardinality of $\Sigma(i)$ will be very large. In fact, for each $i \in N_-$, the number of basic arithmetic operations of $S_i^\star(A)$ is $[2^{n-1} - 1]n + [2^{n-2} \cdot (n + 1) - n](n + 3) = O(2^{n-2}n^2)$ (thus requiring a complexity of $\sum_{S\in\Sigma(i)}[n + (n + 1)|\overline{S}|]$ additions and subtractions and $2\sum_{S\in\Sigma(i)} |\overline{S}|$ multiplications of

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numbers). Therefore, finding some effective criteria with less computational cost to identify CKV-type matrices is interesting.

In this paper, a new simple interesting criterion for CKV-type matrices is obtained, and some possible sparsity patterns in CKV-type matrices are also discussed. In Section 2, we give an equivalent condition for a CKV-type matrix, and then we propose a direct algorithm for identifying CKV-type matrices. In Section 3, we address three possible sparsity patterns in CKV-type matrices, and it is shown that the S-Sparse Ostrowski-Brauer (S-SOB) class provided by Kolotilina in [18] belongs to the CKV-type class. Two new subclasses of CKV-type matrices called S-Sparse Ostrowski-Brauer type-I (S-SOB type-I) and type-II (S-SOB type-II) matrices are presented, which also involve sparsity patterns but are different from S-SOB matrices. Further, a necessary and sufficient condition involving the sparsity pattern for CKV-type matrices is obtained, and a direct algorithm that requires less computational cost for identifying CKV-type matrices is proposed. Moreover, an alternative eigenvalue localization set involving the sparsity pattern for matrices is presented. Numerical examples are also provided to illustrate the effectiveness of the proposed algorithms. Finally, in Section 4, we give some conclusions to end this paper.

2. An algorithm for identifying CKV-type matrices. We start with some preliminaries and definitions. Let $\mathbb{Z}^{n\times n}$ be the set of all matrices $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ with $a_{ij} \leq 0$ for all $i \neq j$. A matrix $A \in \mathbb{Z}^{n\times n}$ is a nonsingular M-matrix if its inverse is nonnegative, i.e., $A^{-1} \geq 0$ [2]. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a strictly diagonally dominant (SDD) matrix [20] if $|a_{ii}| > r_i(A)$ for all $i \in N$, where $r_i(A) = \sum_{j\in N\setminus\{i\}} |a_{ij}|$.

The following simple lemma is needed.

LEMMA 2.1. *Let $a \geq b \geq 0$, $a > c \geq 0$, $e \geq f > 0$, and $d \geq 0$. Then,*

$$\frac{b-c}{a-c} \leq \frac{b}{a} \qquad \textit{and} \qquad \frac{f}{e} \leq \frac{f+d}{e+d}.$$

Proof. By simple computation,

$$\frac{b-c}{a-c} - \frac{b}{a} = \frac{c(b-a)}{a(a-c)} \leq 0 \qquad \text{and} \qquad \frac{f}{e} - \frac{f+d}{e+d} = \frac{d(f-e)}{e(e+d)} \leq 0.$$

This completes the proof. ∎

We next give a necessary condition for a CKV-type matrix.

THEOREM 2.2. *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a CKV-type matrix with $N_- \neq \emptyset$ (i.e., there exists $S \in S_i^\star(A)$ for each $i \in N_-$), then $N_- \cup \Theta_i \in S_i^\star(A)$ for each $i \in N_-$, where*

$$\Theta_i := \left\{ j \in S \setminus N_- : \big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) \leq r_i^{\overline{S}}(A) r_j^S(A) \right\}. \tag{2.1}$$

Proof. Since A is a CKV-type matrix, it follows from Definition 1.1 that for each $i \in N_-$, $S_i^\star(A)$ is not empty, that is, there exists a set $S \in S_i^\star(A)$ such that

$$|a_{ii}| > r_i^S(A), \tag{2.2}$$

and for all $j \in \overline{S}$,

$$\big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > r_i^{\overline{S}}(A) r_j^S(A). \tag{2.3}$$

By (2.3), we have $N_- \subseteq S$. Otherwise, if there exists an index $j \in N_-$ but $j \notin S$, then $j \in \overline{S}$. This is a contradiction to (2.3) since $|a_{jj}| \leq r_j(A)$ and $|a_{ii}| \leq r_i(A)$. Then,

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$S = N_- \cup (S \setminus N_-)$. If $S \setminus N_- = \emptyset$, then $\Theta_i = \emptyset$, and thus the conclusion follows from (2.2) and (2.3). We next consider the case $S \setminus N_- \neq \emptyset$.

If $\Theta_i = \emptyset$, then from (2.1) we have that for all $j \in S \setminus N_-$ inequality (2.3) holds. Let

$$N_+ := \{i \in N : |a_{ii}| > r_i(A)\}.$$

Note that $N_+ = (S \setminus N_-) \cup \overline{S}$. Then, for each $i \in N_-$, by Lemma 2.1 and (2.3), it holds that for $j \in N_+$,

$$\begin{aligned}\frac{r_j^{N_-}(A)}{|a_{jj}| - r_j^{N_+}(A)} &\le \frac{r_j^{N_-}(A) + r_j^{S\setminus N_-}(A)}{|a_{jj}| - r_j^{N_+}(A) + r_j^{S\setminus N_-}(A)} = \frac{r_j^{S}(A)}{|a_{jj}| - r_j^{\overline{S}}(A)} \\ &< \frac{|a_{ii}| - r_i^{S}(A)}{r_i^{\overline{S}}(A)} = \frac{|a_{ii}| - r_i^{N_-}(A) - r_i^{S\setminus N_-}(A)}{r_i^{N_+}(A) - r_i^{S\setminus N_-}(A)} \le \frac{|a_{ii}| - r_i^{N_-}(A)}{r_i^{N_+}(A)},\end{aligned}$$

which together with

$$|a_{ii}| - r_i^{N_-}(A) - r_i^{S\setminus N_-}(A) = |a_{ii}| - r_i^{S}(A) > 0$$

implies that $N_- \in S_i^\star(A)$.

If $\Theta_i \neq \emptyset$, then from (2.1) we have that for all $j \in (S \setminus N_-) \setminus \Theta_i$ inequality (2.3) holds. Note that

$$S = N_- \cup (S \setminus N_-) = (N_- \cup \Theta_i) \cup ((S \setminus N_-) \setminus \Theta_i)$$

and $N \setminus (N_- \cup \Theta_i) = ((S \setminus N_-) \setminus \Theta_i) \cup \overline{S}$. Then, for each $i \in N_-$, it follows from Lemma 2.1, (2.2), and (2.3) that

$$|a_{ii}| - r_i^{N_- \cup \Theta_i}(A) \ge |a_{ii}| - r_i^{S}(A) > 0,$$

and for all $j \in N \setminus (N_- \cup \Theta_i)$,

$$\begin{aligned}\frac{r_j^{N_- \cup \Theta_i}(A)}{|a_{jj}| - r_j^{N\setminus(N_- \cup \Theta_i)}(A)} &= \frac{r_j^{S}(A) - r_j^{(S\setminus N_-)\setminus\Theta_i}(A)}{|a_{jj}| - r_j^{\overline{S}}(A) - r_j^{(S\setminus N_-)\setminus\Theta_i}(A)} \\ &\le \frac{r_j^{S}(A)}{|a_{jj}| - r_j^{\overline{S}}(A)} < \frac{|a_{ii}| - r_i^{S}(A)}{r_i^{\overline{S}}(A)} \\ &\le \frac{|a_{ii}| - r_i^{S}(A) + r_i^{(S\setminus N_-)\setminus\Theta_i}(A)}{r_i^{\overline{S}}(A) + r_i^{(S\setminus N_-)\setminus\Theta_i}(A)} = \frac{|a_{ii}| - r_i^{N_- \cup \Theta_i}(A)}{r_i^{N\setminus(N_- \cup \Theta_i)}(A)}.\end{aligned}$$

This means that $N_- \cup \Theta_i \in S_i^\star(A)$ for $i \in N_-$. This completes the proof. ◻

The following is a sufficient condition such that A is not a CKV-type matrix.

THEOREM 2.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ with $N_- \neq \emptyset$, and let Δ be a nonempty subset of $N_- \cup \Theta_i$ with Θ_i given by Theorem 2.2. If $|a_{ii}| \le r_i^{\Delta}(A)$ for some $i \in N_-$, then A is not a CKV-type matrix.*

Proof. Suppose, on the contrary, that A is a CKV-type matrix. By Definition 1.1, it follows that for each $i \in N_-$, there is a set $S \in S_i^\star(A)$ such that (2.2) and (2.3) hold. Note that $N_- \subseteq S$. We next divide our proof into two cases.

Case I. If $\Delta = N_-$, then it follows from (2.2) that

$$|a_{ii}| - r_i^{N_-}(A) - r_i^{S\setminus N^-}(A) = |a_{ii}| - r_i^{S}(A) > 0,$$

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which gives that $|a_{ii}| - r_i^{N_-}(A) > 0$. This is a contradiction to $|a_{ii}| \leq r_i^{N_-}(A)$ for some $i \in N_-$.

Case II. If $\Delta \neq N_-$, then by $N_- \cup \Theta_i \subseteq S$, we obtain that $\Delta \subseteq S$. Hence, for each $i \in N_-$, it follows from (2.2) that

$$|a_{ii}| - r_i^{\Delta}(A) \geq |a_{ii}| - r_i^{\Delta}(A) - r_i^{S\setminus\Delta}(A) = |a_{ii}| - r_i^{S}(A) > 0.$$

This is a contradiction to $|a_{ii}| \leq r_i^{\Delta}(A)$ for some $i \in N_-$. From Case I and Case II, the conclusion follows. $\square$

By Theorems 2.2 and 2.3, we give a programmable criteria for identifying CKV-type matrices. Before that, we present Algorithm 1 that will be used later.

Algorithm 1 A method for generating sets $S_i^{(k)}$ for each $i \in N_-$.

Input. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ with $N_- \neq \emptyset$. Set $m := |N_+|$.
Step 0. For each $i \in N_-$, $k = 0$, $S_i^{(0)} := N_-$.
Step 1. Compute

$$\Theta_i^{(k)} := \left\{ j \in \overline{S_i^{(k)}} : \left(|a_{ii}| - r_i^{S_i^{(k)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(k)}}}(A)\right) \leq r_i^{\overline{S_i^{(k)}}}(A) r_j^{S_i^{(k)}}(A) \right\}.$$

Step 2. If $\Theta_i^{(k)} \neq \emptyset$, then $k = k+1$, $S_i^{(k)} := S_i^{(k-1)} \cup \Theta_i^{(k-1)}$, and go to step 1. Otherwise, stop, output $S_i^{(k)}$ for some $0 \leq k \leq m$.

Output. $S_i^{(0)} = N_-$ or $S_i^{(k)} = N_- \cup \left(\bigcup_{j=0}^{k-1} \Theta_i^{(j)} \right)$ for some $1 \leq k \leq m$.

THEOREM 2.4. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a CKV-type matrix if and only if $N_- = \emptyset$ or for each $i \in N_-$, there exists $S_i^{(k)}$ generated by Algorithm 1 such that $S_i^{(k)} \in S_i^\star(A)$.*

Proof. Sufficiency obviously holds. We only need to prove the necessity. Suppose that A is a CKV-type matrix. If A is an SDD matrix, then $N_- = \emptyset$. If A is not an SDD matrix, then it follows from Definition 1.1 that for each $i \in N_-$, there is a set $S \in S_i^\star(A)$ such that (2.3) holds, and by Theorem 2.2, $N_- \cup \Theta_i \in S_i^\star(A)$, where Θ_i is given by (2.1).

We next show that $S_i^{(k)}$ generated by Algorithm 1 is a subset of $N_- \cup \Theta_i$, i.e., that we have $S_i^{(k)} \subseteq (N_- \cup \Theta_i)$. Note that $S_i^{(0)} = N_-$ and $S_i^{(k)} = N_- \cup (\bigcup_{j=0}^{k-1} \Theta_i^{(j)})$ for some $1 \leq k \leq |N_+|$. Then, we only have to prove that $\bigcup_{j=0}^{k-1} \Theta_i^{(j)} \subseteq \Theta_i$. To this end, we will prove that $\Theta_i^{(j)} \subseteq \Theta_i$, for $j = 0, 1, \ldots, k-1$.

(i) Proving $\Theta_i^{(0)} \subseteq \Theta_i$. It follows from (2.3) that for all $j \in \overline{S}$,

$$\left(|a_{ii}| - r_i^{N_-}(A)\right)\left(|a_{jj}| - r_j^{N_+}(A)\right) > r_i^{N_+}(A) r_j^{N_-}(A). \tag{2.4}$$

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In fact, if for some $j \in \overline{S}$, (2.4) does not hold, then from Lemma 2.1 and $N_- \subseteq S$, we have

$$\begin{aligned}
\frac{|a_{ii}| - r_i^S(A)}{r_i^{\overline{S}}(A)} &= \frac{|a_{ii}| - r_i^{N_-}(A) - r_i^{S\setminus N_-}(A)}{r_i^{N_+}(A) - r_i^{S\setminus N_-}(A)} \leq \frac{|a_{ii}| - r_i^{N_-}(A)}{r_i^{N_+}(A)} \\
&\leq \frac{r_j^{N_-}(A)}{|a_{jj}| - r_j^{N_+}(A)} \leq \frac{r_j^{N_-}(A) + r_j^{S\setminus N_-}(A)}{|a_{jj}| - r_j^{N_+}(A) + r_j^{S\setminus N_-}(A)} \\
&= \frac{r_j^S(A)}{|a_{jj}| - r_j^{\overline{S}}(A)},
\end{aligned} \tag{2.5}$$

which contradicts (2.3). Note that $N_+ = (S \setminus N_-) \cup \overline{S}$. Hence, from (2.4) and Algorithm 1 we have

$$\begin{aligned}
\Theta_i^{(0)} &= \left\{ j \in N_+ : \left(|a_{ii}| - r_i^{N_-}(A)\right)\left(|a_{jj}| - r_j^{N_+}(A)\right) \leq r_i^{N_+}(A) r_j^{N_-}(A) \right\} \\
&= \left\{ j \in S \setminus N_- : \left(|a_{ii}| - r_i^{N_-}(A)\right)\left(|a_{jj}| - r_j^{N_+}(A)\right) \leq r_i^{N_+}(A) r_j^{N_-}(A) \right\},
\end{aligned}$$

which together with (2.1) and (2.5) imply that $\Theta_i^{(0)} \subseteq \Theta_i$.

(ii) Proving $\Theta_i^{(1)} \subseteq \Theta_i$. By Algorithm 1, we know that $S_i^{(1)} = N_- \cup \Theta_i^{(0)}$. It follows from (2.3) that for all $j \in \overline{S}$,

$$\left(|a_{ii}| - r_i^{S_i^{(1)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(1)}}}(A)\right) > r_i^{\overline{S_i^{(1)}}}(A) r_j^{S_i^{(1)}}(A). \tag{2.6}$$

In fact, if for some $j \in \overline{S}$, (2.6) does not hold, then according to Lemma 2.1 and the fact that $(N_- \cup \Theta_i^{(0)}) \subseteq S$, we have

$$\begin{aligned}
\frac{|a_{ii}| - r_i^S(A)}{r_i^{\overline{S}}(A)} &= \frac{|a_{ii}| - r_i^{N_-\cup\Theta_i^{(0)}}(A) - r_i^{S\setminus(N_-\cup\Theta_i^{(0)})}(A)}{r_i^{N_+\setminus\Theta_i^{(0)}}(A) - r_i^{S\setminus(N_-\cup\Theta_i^{(0)})}(A)} \leq \frac{|a_{ii}| - r_i^{N_-\cup\Theta_i^{(0)}}(A)}{r_i^{N_+\setminus\Theta_i^{(0)}}(A)} \\
&\leq \frac{r_j^{N_-\cup\Theta_i^{(0)}}(A)}{|a_{jj}| - r_j^{N_+\setminus\Theta_i^{(0)}}(A)} \leq \frac{r_j^{N_-\cup\Theta_i^{(0)}}(A) + r_j^{S\setminus(N_-\cup\Theta_i^{(0)})}(A)}{|a_{jj}| - r_j^{N_+\setminus\Theta_i^{(0)}}(A) + r_j^{S\setminus(N_-\cup\Theta_i^{(0)})}(A)} \\
&= \frac{r_j^S(A)}{|a_{jj}| - r_j^{\overline{S}}(A)},
\end{aligned} \tag{2.7}$$

which contradicts (2.3). Note that

$$\overline{S_i^{(1)}} = N \setminus (N_- \cup \Theta_i^{(0)}) = N_+ \setminus \Theta_i^{(0)} = (S \setminus (N_- \cup \Theta_i^{(0)})) \cup \overline{S}.$$

Then, from (2.6) and Algorithm 1, it holds that

$$\begin{aligned}
\Theta_i^{(1)} &= \left\{ j \in \overline{S_i^{(1)}} : \left(|a_{ii}| - r_i^{S_i^{(1)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(1)}}}(A)\right) \leq r_i^{\overline{S_i^{(1)}}}(A) r_j^{S_i^{(1)}}(A) \right\} \\
&= \Big\{ j \in S \setminus (N_- \cup \Theta_i^{(0)}) : \\
&\qquad\qquad \left(|a_{ii}| - r_i^{S_i^{(1)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(1)}}}(A)\right) \leq r_i^{\overline{S_i^{(1)}}}(A) r_j^{S_i^{(1)}}(A) \Big\},
\end{aligned}$$

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which together with (2.1) and (2.7) imply that $\Theta_i^{(1)} \subseteq \Theta_i$.

Similarly to the proof of (i) and (ii), we can prove that $\Theta_i^{(j)} \subseteq \Theta_i$, for $j = 2, \ldots, k-1$. Hence, $\bigcup\limits_{j=0}^{k-1} \Theta_i^{(j)} \subseteq \Theta_i$, and consequently $S_i^{(k)} \subseteq (N_- \cup \Theta_i)$. This completes the proof. ∎

REMARK 2.5. According to Theorem 2.4, it follows that

(i) a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is not a CKV-type matrix if and only if $N_- \neq \emptyset$ and for some $i \in N_-$, $S_i^{(k)} \notin S_i^\star(A)$;

(ii) if $|a_{ii}| - r_i^{S_i^{(k)}}(A) \leq 0$ for some $i \in N_-$, then A is not a CKV-type matrix, where $S_i^{(k)}$ is given by Algorithm 1.

Next, we present Algorithm 2 for identifying CKV-type matrices on the basis of the above results.

Algorithm 2 A direct method for identifying CKV-type matrices.

Input. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ with $N_- \neq \emptyset$.
Step 0. Compute N_-, and set $m := |N_+|$, $N_- =: \{i_1, i_2, \ldots, i_l\}$, where $l = |N_-|$.
Step 1. For $i \in N_-$, set $k = 0$, $S_i^{(k)} = N_-$, $\overline{S_i^{(k)}} = N_+$, and go to Step 2.
Step 2. If $\overline{S_i^{(k)}} = \emptyset$, then '$A$ is not a CKV-type matrix', stop. Otherwise, compute

$$d_i^{(k)} := |a_{ii}| - r_i^{S_i^{(k)}}(A).$$

If $d_i^{(k)} \leq 0$, then 'A is not a CKV-type matrix', stop. Otherwise, compute

$$\Theta_i^{(k)} := \left\{ j \in \overline{S_i^{(k)}} : \left(|a_{ii}| - r_i^{S_i^{(k)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(k)}}}(A)\right) \leq r_i^{\overline{S_i^{(k)}}}(A) r_j^{S_i^{(k)}}(A) \right\}.$$

If $\Theta_i^{(k)} = \emptyset$, then $S_i^{(k)} \in S_i^\star(A)$, i.e., $S_i^\star(A) \neq \emptyset$. Otherwise, go to Step 3.
Step 3. Set

$$S_i^{(k+1)} := S_i^{(k)} \cup \Theta_i^{(k)} \text{ and } \overline{S_i^{(k+1)}} := \overline{S_i^{(k)}} \setminus \Theta_i^{(k)},$$

and go to Step 2 (Replace k by $k+1$).
Output. A is either not a CKV-type matrix or a CKV-type matrix for $\{S_{i_1}^{(k_{i_1})}, S_{i_2}^{(k_{i_2})}, \ldots, S_{i_l}^{(k_{i_l})}\}$, where $k_{i_t} \in \{0, 1, \ldots, m\}$ with $t = 1, 2, \ldots, l$.

REMARK 2.6.

(i) Algorithm 2 is a direct method for identifying CKV-matrices, and the calculations only depend on the elements of the involved matrix and the subsets of N. Therefore, Algorithm 2 stops after a finite number of steps.

(ii) The validity of Algorithm 2 follows from Theorem 2.3 and Theorem 2.4. Let us count the computational effort in the individual steps of the iteration in Step 2. Denote

$$\underbrace{|a_{ii}| - r_i^{S_i^{(k)}}(A)}_{①}, \underbrace{|a_{jj}| - r_j^{\overline{S_i^{(k)}}}(A)}_{②}, \underbrace{r_i^{\overline{S_i^{(k)}}}(A)}_{③}, \underbrace{r_j^{S_i^{(k)}}(A)}_{④}.$$

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By computations, ①, ②, ③, and ④, respectively, need $n - |\overline{S_i^{(k)}}|$, $|\overline{S_i^{(k)}}|$, $|\overline{S_i^{(k)}}|$, and $n - |\overline{S_i^{(k)}}|$ additions and subtractions; ① × ② and ③ × ④ for all $j \in \overline{S}$ need $2|\overline{S_i^{(k)}}|$ multiplications; ① × ② − ③ × ④ for all $j \in \overline{S}$ need $|\overline{S_i^{(k)}}|$ subtractions. Hence, the total effort is $n + (n+3)|\overline{S_i^{(k)}}| = O(n^2)$. This implies that the number of basic arithmetic operations of Step 2 and Step 3 is less than $\sum\limits_{k=0}^{m-1}\left[n + (n+3)|\overline{S_i^{(k)}}|\right]$, which can be bounded above by

$$nm + (n+3)\frac{m(m+1)}{2} < n^2 + (n+3)\frac{n(n+1)}{2} = O(n^3).$$

Thus, condition $\Theta_i^{(k)}$ of Algorithm 2 can be verified in polynomial time. Note that $\Sigma(i) := \{S \subsetneq N : i \in S\}$. Then, for each $i \in N_-$, the number of basic arithmetic operations of Definition 1.1 is

$$\begin{aligned}\sum_{S\in\Sigma(i)}\left[n + (n+3)|\overline{S}|\right] &= \sum_{S\in\Sigma(i)} n + \sum_{S\in\Sigma(i)} (n+3)|\overline{S}| \\ &= n(2^{n-1}-1) + (n+3)[2^{n-2}(n+1) - n] = O(2^{n-2}n^2).\end{aligned}$$

Obviously, for a large matrix, the computational cost of Algorithm 2 is much less than that of Definition 1.1.

(iii) Observe from Algorithm 2 that if A is a CKV-type matrix, then the corresponding S for each $i \in N_-$ can be easily obtained instead of traversing all subsets of N.

In the following, we implement Algorithm 2 to show that the identification of Algorithm 2 is made efficiently. We implement all experiments in MATLAB version R2016a by using a PC with 3.40-GHz processors and 64 GB of memory. The MATLAB code for Algorithm 2 is given in Appendix A.

EXAMPLE 2.7. Consider the following matrices arising from the solution of linear systems [21], the finite difference method for free boundary problems [27], and the error control analysis of linear complementary problems [11, 26]:

1.)

$$A_1 = \begin{bmatrix} b+\alpha\sin(\frac{1}{n}) & c & & & \\ a & b+\alpha\sin(\frac{2}{n}) & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b+\alpha\sin(\frac{n-1}{n}) & c \\ & & & a & b+\alpha\sin(1) \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where $b = 2$, $a = c = -1$, and $\alpha = 0$ [27];

2.)

$$A_2 = \begin{bmatrix} 1+m\lambda\theta & -\lambda\theta & & & \\ -\lambda\theta & 1+m\lambda\theta & -\lambda\theta & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda\theta & 1+m\lambda\theta & -\lambda\theta \\ & & & -\lambda\theta & 1+m\lambda\theta \end{bmatrix} \in \mathbb{R}^{n-1\times n-1},$$

where $m > 0$, $\theta = \frac{1}{2}$, and $\lambda = \frac{\Delta\tau}{(\Delta x)^2}$, with $\Delta x = \frac{b-a}{n}$, $\Delta\tau = \frac{1}{2}\frac{\sigma^2 T}{v_{max}}$, $\sigma > 0$, $T > 0$, $v_{max} > 0$ [11];

3.)

$$A_3 = \begin{bmatrix} F & & & & & & & \\ -E & F & & & & & & \\ -E & -E & \widetilde{F} & & & & & \\ & -E & -E & \widetilde{F} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -E & -E & \widetilde{F} & & \\ & & & & -E & -E & \widetilde{F} & \\ & & & & & -E & -E & \widetilde{F} \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where E is the identity matrix of order m, $F = \mathrm{tridiag}(-1, 3, -1) \in \mathbb{R}^{m\times m}$, and $\widetilde{F} = \mathrm{tridiag}(-1, 5, -1) \in \mathbb{R}^{m\times m}$ [26];

4.)

$$A_4 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{200\times 200},$$

where

$$A_{11} = \begin{bmatrix} a_{11} & -0.6 & & & & \\ -0.6 & a_{22} & -0.6 & & & \\ & -0.6 & a_{33} & -0.6 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -0.6 & a_{99,99} & -0.6 \\ & & & & -0.6 & a_{100,100} \end{bmatrix} \in \mathbb{R}^{100\times 100},$$

$$A_{12} = A_{21}^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0.6 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0.6 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{100\times 100},$$

and

$$A_{22} = \begin{bmatrix} 51\times 1.2 & -0.6 & & & & \\ -0.6 & 52\times 1.2 & -0.6 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -0.6 & 148\times 1.2 & -0.6 & \\ & & & -0.6 & 149\times 1.2 & -0.6 \\ & & & & -0.6 & 12000 \end{bmatrix} \in \mathbb{R}^{100\times 100},$$

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with $a_{ii}, i = 1, 2, \dots, 100$, are randomly generated as follows [21]:

$$
\begin{aligned}
a_{ii} = \\
&0.8563, 1.1207, 1.7794, 1.1160, 1.6647, 1.1313, 1.6322, 1.0899, 1.7644, 1.1981,\\
&1.2040, 1.0541, 1.8260, 1.0995, 1.9415, 1.1988, 1.2066, 1.1853, 1.8597, 1.1955,\\
&1.8139, 0.9828, 1.9977, 1.1733, 1.3821, 1.1094, 1.9356, 1.1054, 1.7136, 1.1747,\\
&1.2843, 1.1523, 1.4145, 1.1334, 1.8111, 1.1915, 1.8444, 1.1645, 1.2834, 1.1711,\\
&1.5758, 1.1687, 1.3752, 1.1799, 1.9382, 1.1051, 1.4563, 1.0458, 1.8860, 1.1160,\\
&1.4079, 1.0901, 1.9025, 1.1845, 1.3506, 1.0554, 1.8074, 1.1904, 1.2254, 1.0797,\\
&1.7139, 1.1100, 1.6535, 1.1684, 1.5011, 1.1635, 1.3700, 1.0720, 1.8337, 1.1950,\\
&1.3164, 1.1949, 1.5913, 1.1981, 1.2103, 1.1424, 1.3493, 1.1440, 1.5882, 1.0673,\\
&1.8706, 1.1995, 1.3128, 1.0596, 1.7858, 1.1444, 1.7529, 1.1937, 1.2276, 1.0810,\\
&1.5911, 1.1268, 1.9771, 1.1603, 1.2900, 1.0379, 1.7946, 1.0290, 1.7108, 1.1737.
\end{aligned}
$$

We determine whether they are CKV-type matrices or not by using Algorithm 2. The numerical results are reported in Table 2.1. In this table, 'Yes' and 'No' means the matrix is a CKV-type matrix or not a CKV-type matrix, respectively. It can be seen from Table 2.1 that, for a larger matrix A, Algorithm 2 can easily verify whether A is a CKV-type matrix or not, and when A is a CKV-type matrix, it can output $S \in S_i^\star(A)$ exactly for each $i \in N_-$ instead of traversing all subsets of N. In particular, if there is a common set $S \in S_i^\star(A)$ for all $i \in N_-$ (see the matrix A_3 in Table 2.1), then it can also be concluded that A is an S-SDD matrix. This means that based on Algorithm 2 it is possible to give a direct algorithm to verify the CKV-matrix. This also shows that our numerical results are reasonable and efficient.

TABLE 2.1
Identifying whether a matrix is a CKV-type matrix or not by Algorithm 2.

Order	Matrix	N_-	$S_i^\star(A)$ $(i \in N_-)$	Yes or No
100	A_1	{2,3,...,99}	$S_3^\star(A) = \emptyset$	No
400	A_2	{2,3,...,399}	$S_3^\star(A) = \emptyset$	No
400	A_3	{22,23,...,39}	$\{22, 23, \dots, 39\} \in S_i^\star(A)$	Yes
200	A_4	{1,2,4,...,100}	$\left\{\begin{array}{l} \{1, 2, \dots, 100\} \in S_1^\star(A), \\ N_- \cup \alpha \cup \{25\} \in S_i^\star(A), i = 2, 98, \\ N_- \cup \{11, 17, 31, 39, 59, 75, 89, 95\} \in S_i^\star(A), \\ \qquad i = 4, 14, 26, 28, 46, 50, 62, \\ N_- \cup \{11, 17, 59, 75, 89\} \in S_i^\star(A), \\ \qquad i = 6, 18, 32, 34, 38, 40, 42, 64, 66, 78, 86, 92, 94, \\ N_- \cup \{11, 17, 31, 39, 59, 71, 75, 83, 89, 95\} \in S_i^\star(A), \\ \qquad i = 8, 52, 68, 76, 80, \\ N_- \in S_i^\star(A), i = 10, 16, 74, 82, \\ N_- \cup \alpha \cup \{25, 33, 41, 47, 51, 65\} \in S_i^\star(A), i = 12, 56, \\ N_- \cup \{11\} \in S_i^\star(A), i = 20, 70, 72, 88, \\ N_- \cup \alpha \cup \{25, 33, 47, 51\} \in S_i^\star(A), i = 22, 60, 90, \\ N_- \cup \{11, 17, 59, 75\} \in S_i^\star(A), i = 24, 30, 100, \\ N_- \cup \{11, 17\} \in S_i^\star(A), i = 36, 58, \\ N_- \cup \{11, 17, 75\} \in S_i^\star(A), i = 44, 54, \\ N_- \cup \alpha \in S_{48}^\star(A), \\ N_- \cup \alpha \cup \{25, 33, 47, 51, 65\} \in S_{84}^\star(A), \\ N_- \cup \alpha \cup \{7, 25, 33, 41, 47, 51, 65, 73, 79, 91\} \in S_{96}^\star(A). \end{array}\right.$	Yes

Note: The parameters for A_2 are given as follows: $m = 1.7, a = -1, b = 0.4, \sigma = 0.3, T = 5, \theta = 2$, $v_{max} = 160$, and $\alpha := \{11, 17, 31, 39, 43, 55, 59, 67, 71, 75, 77, 83, 89, 95\}$ in matrix A_4.

Observe that a CKV-type matrix is a nonsingular H-matrix. This means that Algorithm 2 is provided to determine some subclasses of H-matrices, consequently, H-matrices. It is known that some iterative algorithms such as Algorithm AH [1] and Algorithm YZ [29] have been developed for identifying H-matrices. Algorithm AH was presented in [1] to determine the H-matrix characterization of a given irreducible matrix, while Algorithm YZ in [29] can determine the H-matrix characterization for any given matrix. To make a comparison, we consider the matrices generated by Table 2.2, where A_5 is a randomly generated matrix and A_6 is a banded matrix arising from the convergence analysis of modulus-based methods for linear complementary problems [26]. Some parameters are given as follows: maxit $= 10000$, $\varepsilon = 10^{-8}$, $\delta = 10^{-10}$. We list the numerical results in Table 2.3, where iter and CPU (s) denote the number of iterations and the elapsed CPU time in seconds, respectively.

TABLE 2.2
Generating matrices A_5 and A_6.

```
A_5=rand(n)+55*eye(n).

function A_6=lcp(n)
T=zeros(n)-0.5*diag(ones(n-1,1),1)+0.5*diag(ones(n-2,1),-2);
E=eye(n);E(1,end)=-4;E(end,1)=-3;S=10*eye(n)+T;
A=S;P=E;Q=E;
for j=1:(n-1)
  A=blkdiag(A,S);P=blkdiag(P,E);Q=blkdiag(Q,E);
end
P(:,(n^2-n+1):end)=[];P=[zeros(n^2,n),P];
Q(:,1:n)=[];Q=[Q,zeros(n^2,n)];
A=A+P+Q.
```

TABLE 2.3
The numerical results for matrices A_5 and A_6.

Order	Matrix	Algorithm	iter	Result	CPU(s)
100	A_5	AH	2	A_5 is an H-matrix	0.005849
		YZ	1		0.003216
		Algorithm 2	$--$		0.001072
100	A_6	AH	2	A_6 is an H-matrix	0.004097
		YZ	1		0.004064
		Algorithm 2	$--$		0.002638
900	A_6	AH	2	A_6 is an H-matrix	0.382535
		YZ	1		0.190917
		Algorithm 2	$--$		0.056807
2500	A_6	AH	2	A_6 is an H-matrix	8.005067
		YZ	1		1.842606
		Algorithm 2	$--$		0.377416

From Table 2.3, we see that Algorithms 2, AH, and YZ are all effective, and the numerical results indicate that Algorithm 2 performs better than Algorithms AH and YZ, especially for larger matrices.

3. Possible sparsity patterns in CKV-type matrices.

3.1. S-SOB matrices. Very recently, by considering the sparsity structure of the matrix, an interesting new subclass of nonsingular H-matrices named S-Sparse Ostrowski-Brauer (S-SOB) matrices have been discovered by Kolotilina in [18].

DEFINITION 3.1. *Let S be a nonempty subset of N and $\overline{S} = N \setminus S$. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an S-SOB matrix if*

$$\begin{cases} |a_{ii}| > r_i^S(A), & i \in S, \\ |a_{jj}| > r_j^{\overline{S}}(A), & j \in \overline{S}, \\ (|a_{ii}| - r_i^S(A))|a_{jj}| > r_i^{\overline{S}}(A) r_j(A), & i \in S, j \in \overline{S} \textit{ satisfying } a_{ij} \neq 0, \\ (|a_{jj}| - r_j^{\overline{S}}(A))|a_{ii}| > r_j^S(A) r_i(A), & i \in S, j \in \overline{S} \textit{ satisfying } a_{ji} \neq 0, \end{cases}$$

where $r_i^S(A) := \sum_{j \in S\setminus\{i\}} |a_{ij}|$.

The following result shows that the S-SOB class belongs to the CKV-type class.

PROPOSITION 3.2. *If A is an S-SOB matrix, then A is a CKV-type matrix, that is,*

$$\{S\text{-}SOB\} \subseteq \{CKV\text{-}type\}.$$

Proof. Note that $N_- := \{i \in N : |a_{ii}| \leq r_i(A)\}$. If $N_- = \emptyset$, then A is an SDD matrix. For this case, the conclusion is obviously true. If $N_- \neq \emptyset$, then for any $i_0 \in N_-$, we have $i_0 \in S \cup \overline{S}$, where S is a nonempty subset of N such that A is an S-SOB matrix.

The first case. If $i_0 \in S$, then since A is an S-SOB matrix, it follows that

$$|a_{i_0,i_0}| > r_{i_0}^S(A) \quad \text{and} \tag{3.1}$$

$$(|a_{i_0,i_0}| - r_{i_0}^S(A))|a_{jj}| > r_{i_0}^{\overline{S}}(A) r_j(A) \quad \text{for all } j \in \overline{S} \text{ satisfying } a_{i_0,j} \neq 0. \tag{3.2}$$

According to (3.1), it holds that there exists an index $j_0 \in \overline{S}$ such that $a_{i_0,j_0} \neq 0$. Otherwise, if $a_{i_0,j} = 0$ for all $j \in \overline{S}$, then

$$|a_{i_0,i_0}| - r_{i_0}^S(A) = |a_{i_0,i_0}| - r_{i_0}(A) \leq 0,$$

which contradicts (3.1). On the other hand, if $a_{i_0,j} \neq 0$ for some $j \in \overline{S}$, then from $|a_{i_0,i_0}| \leq r_{i_0}(A)$ and (3.2) it holds that $|a_{jj}| > r_j(A)$. Thus, without loss of generality, we assume that $\overline{S} = \{j_1, \ldots, j_k, j_{k+1}, \ldots, j_l\}$ such that $a_{i_0,j} = 0$ for all $j \in \{j_1, \ldots, j_k\}$ and $a_{i_0,j} \neq 0$ for all $j \in \{j_{k+1}, \ldots, j_l\}$, where $1 \leq l < n$.

We now show that A is a CKV-type matrix. Let

$$\overline{S'} := \overline{S} \setminus \{j_1, \ldots, j_k\} = \{j_{k+1}, \ldots, j_l\} \quad \text{and} \quad S' := S \cup \{j_1, \ldots, j_k\}.$$

It follows from (3.1) and (3.2) that

$$|a_{i_0,i_0}| - r_{i_0}^{S'}(A) = |a_{i_0,i_0}| - r_{i_0}^S(A) > 0,$$

and for all $j \in \overline{S'}$,

$$\begin{aligned} &\big(|a_{i_0,i_0}| - r_{i_0}^{S'}(A)\big)\big(|a_{jj}| - r_j^{\overline{S'}}(A)\big) - r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A) \\ &\qquad = \big(|a_{i_0,i_0}| - r_{i_0}^{S'}(A)\big)|a_{jj}| - \big(|a_{i_0,i_0}| - r_{i_0}^{S'}(A)\big) r_j^{\overline{S'}}(A) - r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A) \\ &\qquad \geq \big(|a_{i_0,i_0}| - r_{i_0}^{S'}(A)\big)|a_{jj}| - r_{i_0}^{\overline{S'}}(A) r_j^{\overline{S'}}(A) - r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A) \\ &\qquad = \big(|a_{i_0,i_0}| - r_{i_0}^{S'}(A)\big)|a_{jj}| - r_{i_0}^{\overline{S'}}(A) r_j(A) \\ &\qquad = \big(|a_{i_0,i_0}| - r_{i_0}^{S}(A)\big)|a_{jj}| - r_{i_0}^{\overline{S}}(A) r_j(A) > 0, \end{aligned} \tag{3.3}$$

which implies that $S_{i_0}^\star(A) \neq \emptyset$.

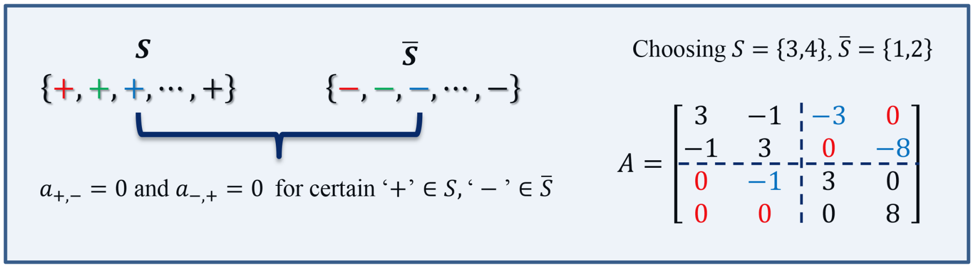

FIG. 3.1. *Sparsity pattern (red zero elements) for a given matrix.*

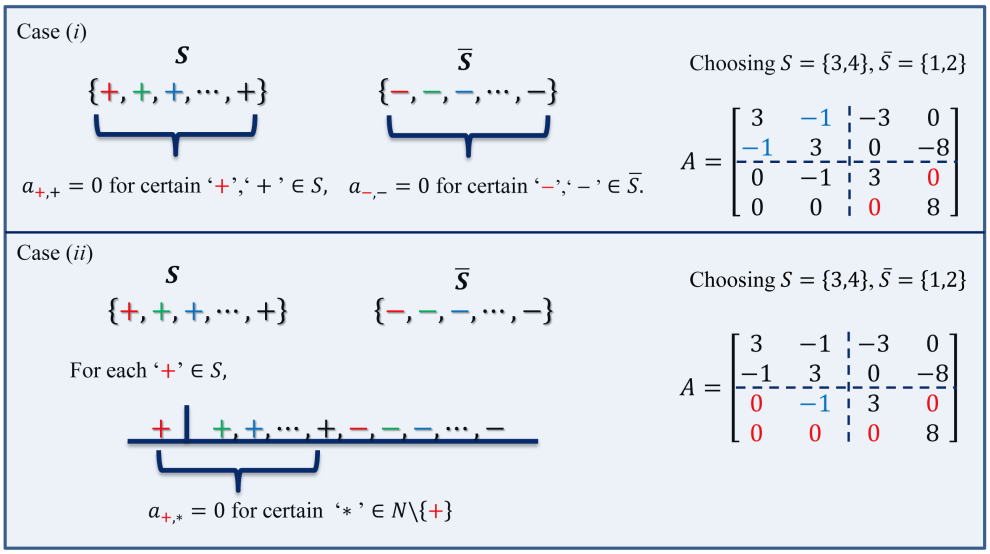

FIG. 3.2. *Other sparsity patterns (red zero elements) for a given matrix.*

The second case. If $i_0 \in \overline{S}$, then similarly to the proof of the first case, we can prove that $S^{\star}_{i_0}(A) \neq \emptyset$.

Hence, from the above two cases, $S^{\star}_{i_0}(A) \neq \emptyset$ for all $i_0 \in N_-$, meaning that A is a CKV-type matrix. The proof is complete. $\square$

3.2. S-SOB type-I matrices and S-SOB type-II matrices. Observe from Definition 3.1 that S-SOB matrices only consider the influence of $a_{ij} = 0$ and $a_{ji} = 0$ for certain $i \in S$, $j \in \overline{S}$ on the non-singularity. The corresponding sparsity pattern (e.g., the position of the red zero elements in matrix A) is illustrated in Figure 3.1, in which the indices belonging to S and $\overline{S}$ are represented by '$+$' and '$-$', respectively, and the matrix A is given in [19]. However, zero elements in other positions in the sparsity pattern might also affect the non-singularity of the matrix such as in

Case 1: $a_{ij} = 0$ for certain $i, j \in S$ or $i, j \in \overline{S}$, $i \neq j$, and

Case 2: for each $i \in S$, $a_{ij} = 0$ for certain $j \in N \setminus \{i\}$; see Figure 3.2.

In this section, we introduce two new classes of matrices, called S-SOB type-I and S-SOB type-II matrices, which involve sparsity patterns corresponding to Case 1 and Case 2, respectively, and prove that they are all subclasses of CKV-type matrices.

3.2.1. S-SOB type-I matrices. In the following, we define a new class of matrices, which is called S-SOB type-I matrices.

DEFINITION 3.3. *Let S be a nonempty subset of N and $\overline{S} = N \setminus S$. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an S-Sparse Ostrowski-Brauer type-I (S-SOB type-I) matrix if*

$$\begin{cases} |a_{ii}| > r_i^{\overline{S}}(A), & i \in S, \\ (|a_{ii}| - r_i^{\overline{S}}(A))|a_{jj}| > r_i^S(A) r_j(A), & i,j \in S,\ i \neq j \text{ satisfying } a_{ij} \neq 0, \end{cases}$$

and

$$\begin{cases} |a_{ii}| > r_i^{S}(A), & i \in \overline{S}, \\ (|a_{ii}| - r_i^{S}(A))|a_{jj}| > r_i^{\overline{S}}(A) r_j(A), & i,j \in \overline{S},\ i \neq j \text{ satisfying } a_{ij} \neq 0. \end{cases}$$

The following result shows that the S-SOB type-I class belongs to the CKV-type class.

PROPOSITION 3.4. *If A is an S-SOB type-I matrix, then A is a CKV-type matrix, that is,*

$$\{S\text{-SOB type-I}\} \subseteq \{\text{CKV-type}\}.$$

Proof. Note that $N_- := \{i \in N : |a_{ii}| \leq r_i(A)\}$. If $N_- = \emptyset$, then A is an SDD matrix. For this case, the conclusion is obviously true. If $N_- \neq \emptyset$, then for any $i_0 \in N_-$, we have $i_0 \in S \cup \overline{S}$, where S is a nonempty subset of N such that A is an S-SOB type-I matrix.

The first case. If $i_0 \in S$, then since A is an S-SOB type-I matrix, it follows that

$$\begin{aligned} |a_{i_0,i_0}| &> r_{i_0}^{\overline{S}}(A) && \text{and} \\ (|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))|a_{jj}| &> r_{i_0}^S(A) r_j(A) && \text{if } a_{i_0,j} \neq 0 \text{ for } j \in S \setminus \{i_0\}. \end{aligned}$$

Without loss of generality, we assume that $S = \{i_0, i_1, \ldots, i_k\}$ and $\overline{S} = \{j_1, \ldots, j_l\}$. Let

$$S' := \{i_0, i_1, \ldots, i_s, j_1, \ldots, j_l\} \qquad \text{and} \qquad \overline{S'} := \{i_{s+1}, \ldots, i_k\},$$

where $a_{i_0,j} = 0$ for all $j \in \{i_1, \ldots, i_s\}$ and $a_{i_0,j} \neq 0$ for all $j \in \overline{S'}$. Then, we easily obtain that

$$|a_{i_0,i_0}| - r_{i_0}^{S'}(A) = |a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A) > 0,$$

and for all $j \in \overline{S'}$,

$$(|a_{i_0,i_0}| - r_{i_0}^{S'}(A))|a_{jj}| = (|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))|a_{jj}| > r_{i_0}^{S}(A) r_j(A) = r_{i_0}^{\overline{S'}}(A) r_j(A). \tag{3.4}$$

According to $|a_{i_0,i_0}| \leq r_{i_0}(A)$, it follows from (3.3) and (3.4) that

$$(|a_{i_0,i_0}| - r_{i_0}^{S'}(A))(|a_{jj}| - r_j^{\overline{S'}}(A)) > r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A).$$

This means that $S' \in S_{i_0}^{\star}(A)$, that is, $S_{i_0}^{\star}(A) \neq \emptyset$.

The second case. If $i_0 \in \overline{S}$, then similarly to the proof of the first case, we can prove that $S_{i_0}^{\star}(A) \neq \emptyset$. From the above two cases, we can conclude that A is a CKV-type matrix. ∎

REMARK 3.5.

(i) For SDD matrices, it follows that $|a_{ii}| > r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A)$, i.e., that $|a_{ii}| - r_i^{\overline{S}}(A) > r_i^S(A)$ for all $i \in N$ and $|a_{jj}| > r_j(A)$ for all $j \in N$, which imply that an SDD matrix is an S-SOB type-I matrix.

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(ii) If $S = N$, then for all $i, j \in N, i \neq j$, we have $r_i^{\overline{S}}(A) = 0$ and $r_i^S(A) = r_i(A)$, and then the conditions of Definition 3.3 reduce to

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A), \qquad i,\ j \in N,\ i \neq j \text{ satisfying } a_{ij} \neq 0,$$

which implies that a DSDD matrix is an S-SOB type-I matrix. Here, a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a doubly strictly diagonally dominant (DSDD) matrix [25] if for all $i, j \in N, j \neq i$,

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A).$$

3.2.2. *S*-SOB type-II matrices. In the following, we define another class of matrices, which is called S-SOB type-II matrices.

DEFINITION 3.6. *Let S be a nonempty subset of N and $\overline{S} = N \setminus S$. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an S-Sparse Ostrowski-Brauer type-II (S-SOB type-II) matrix if for each $j \in S$ and all $i \in N \setminus \{j\}$,*

$$\begin{cases} |a_{ii}| > r_i^{\overline{S}}(A), & \\ (|a_{ii}| - r_i^{\overline{S}}(A))|a_{jj}| > r_i^S(A)r_j(A) & \text{if } a_{ij} \neq 0. \end{cases}$$

A technique similar to the one used in Proposition 3.4, the details of which we report in Appendix B, shows that the S-SOB type-II class also belongs to the CKV-type class.

PROPOSITION 3.7. *If A is an S-SOB type-II matrix, then A is a CKV-type matrix, that is,*

$$\{S\text{-}SOB\ type\text{-}II\} \subseteq \{CKV\text{-}type\}.$$

REMARK 3.8.

(i) Obverse that for any $i, j \in N, i \neq j$, $|a_{ii}| > r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A)$, i.e., $|a_{ii}| - r_i^{\overline{S}}(A) > r_i^S(A)$ and $|a_{jj}| > r_j(A)$. This means that an SDD matrix is an S-SOB type-II matrix for any subset S of N.

(ii) If S is a singleton, that is, $S = \{j\}$, then from the proof of Proposition 3.7 we get that a Dashnic-Zusmanovich (DZ) matrix is an S-SOB type-II matrix. Here, a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a DZ matrix [12] if there exists an index $j \in N$ such that for any $i \in N, i \neq j$,

$$\left(|a_{ii}| - r_i^j(A)\right)|a_{jj}| > |a_{ij}|r_j(A). \tag{3.5}$$

At the end of this section, the relationships among SDD [20], DSDD [25], DZ [12], DZ-type [32], CKV (Σ-SDD), CKV-type, SOB [18], S-SOB, S-SOB type-I, and S-SOB type-II matrices are given (for the details, see Appendix C).

As reported in [8], [19], and [32], the relations of SDD, DSDD, DZ, DZ-type, CKV, and CKV-type matrices are

- $\{$SDD$\} \subseteq \{$DSDD$\} \subseteq \{$DZ$\}$, $\{$SDD$\} \subseteq \{$DZ-type$\}$
- $\{$DSDD$\} \not\subseteq \{$DZ-type$\}$ and $\{$DZ-type$\} \not\subseteq \{$DSDD$\}$
- $\{$DZ$\} \not\subseteq \{$DZ-type$\}$ and $\{$DZ-type$\} \not\subseteq \{$DZ$\}$
- $\{$CKV$\} \not\subseteq \{$DZ-type$\}$ and $\{$DZ-type$\} \not\subseteq \{$CKV$\}$
- $\{$CKV$\} \subseteq \{$CKV-type$\}$ and $\{$DZ-type$\} \subseteq \{$CKV-type$\}$.

According to Remark 3.5, Remark 3.8, and Appendix C, an illustration is given in Figure 3.3 to show the relations among SDD, DSDD, DZ, DZ-type, CKV, CKV-type, SOB, S-SOB, S-SOB type-I, and S-SOB type-II matrices.

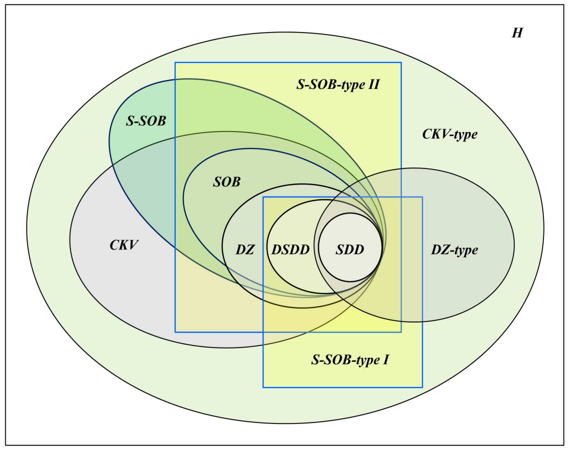

FIG. 3.3. *Relations between some subclasses of H-matrices.*

3.3. A necessary and sufficient condition involving the sparsity pattern for a CKV-type matrix.

THEOREM 3.9. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a CKV-type matrix if and only if $N_- = \emptyset$ or $S_i^{\blacktriangle}(A)$ is not empty for all $i \in N_-$, where*

$$S_i^{\blacktriangle}(A) := \Big\{ S \in \Sigma(i) : N_- \subseteq S,\ |a_{ii}| > r_i^S(A),\ \textit{and for all } j \in \blacktriangle_i,$$
$$\big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > r_i^{\overline{S}}(A) r_j^S(A) \Big\},$$

with N_- and $\Sigma(i)$ being defined as in Definition 1.1 and

$$\blacktriangle_i := \{ j \in \overline{S} : a_{jk} \neq 0 \textit{ for some } k \in S\}.$$

Proof. Note that

$$S_i^{\star}(A) := \Big\{ S \in \Sigma(i) : |a_{ii}| > r_i^S(A),\ \text{and for all } j \in \overline{S}$$
$$\big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > r_i^{\overline{S}}(A) r_j^S(A) \Big\}.$$

By the proof of Theorem 2.2, we know that $N_- \subseteq S$ for each $S \in S_i^{\star}(A)$. Moreover, for each $S \in S_i^{\blacktriangle}(A)$, from the definition of $\blacktriangle_i$, it follows that for all $j \in \overline{S} \setminus \blacktriangle_i$, $a_{jk} = 0$ for all $k \in S$, and thus

$$\big(|a_{ii}| - r_i^S(A)\big)\big(|a_{jj}| - r_j^{\overline{S}}(A)\big) > 0 = r_i^{\overline{S}}(A) r_j^S(A).$$

On the other hand, if $S \in S_i^{\star}(A)$, then $S \in S_i^{\blacktriangle}(A)$. Hence, $S_i^{\blacktriangle}(A) = S_i^{\star}(A)$. This completes the proof. ∎

As shown in [8], by the nonsingularity of CKV-type matrices, a new eigenvalue localization set for matrices has been obtained (see [8, Theorem 16]). In the following, on the basis of Theorem 3.9, we give a new eigenvalue localization set equivalent to that of [8], but it involves the sparsity pattern and requires less computation. The proof of this result is similar to that of [8, Theorem 16] and hence omitted.

THEOREM 3.10. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Denote*

$$\Gamma_i^S(A) := \left\{z \in \mathbb{C} : |z - a_{ii}| \le r_i^S(A)\right\}$$

and

$$V_{ij}^S(A) := \left\{z \in \mathbb{C} : (|z - a_{ii}| - r_i^S(A))(|z - a_{jj}| - r_j^{\overline{S}}(A)) \le r_i^{\overline{S}}(A) r_j^S(A)\right\}.$$

Then for every eigenvalue λ it holds that

$$\lambda \in \mathcal{V}_\lambda^{\blacktriangle}(A) := \bigcup_{i\in N_{\lambda I - A}} \bigcap_{S\in\Sigma(i)} \left(\Gamma_i^S(A) \cup \left(\bigcup_{j\in\blacktriangle_i} V_{ij}^S(A)\right)\right),$$

hence

$$\sigma(A) \subseteq \mathcal{V}^{\blacktriangle}(A) := \bigcup_{i\in N} \bigcap_{S\in\Sigma(i)} \left(\Gamma_i^S(A) \cup \left(\bigcup_{j\in\blacktriangle_i} V_{ij}^S(A)\right)\right),$$

where $\Sigma(i)$ and $\blacktriangle_i$ are defined in Definition 1.1 and Theorem 3.9, respectively.

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 5 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

In the progress of computing the eigenvalue localization set for A using [8, Theorem 16], we need to calculate $V_{ij}^S(A)$ for all $j \in \overline{S}$. However, using Theorem 3.10, we only need to compute $V_{ij}^S(A)$ for some $j \in \overline{S}$ (i.e., $j \in \blacktriangle_i$; see the underlined and bold indicators below) instead of traversing all indices of $\overline{S}$. This shows that Theorem 3.10 can greatly reduce the computational cost of the eigenvalue localization set for sparse matrices. For each $i \in N$, the sets $V_{ij}^S(A)$ for all $S \in \Sigma(i)$ are given as follows.

$$V_{1j}^S(A) \begin{cases} S = \{1\}, & \overline{S} = \{2, \underline{\mathbf{3}}, 4\}, \\ S = \{1,2\}, & \overline{S} = \{\underline{\mathbf{3}}, 4\}, \\ S = \{1,3\}, & \overline{S} = \{2, 4\}, \\ S = \{1,4\}, & \overline{S} = \{2, \underline{\mathbf{3}}\}, \\ S = \{1,2,3\}, & \overline{S} = \{4\}, \\ S = \{1,2,4\}, & \overline{S} = \{\underline{\mathbf{3}}\}, \\ S = \{1,3,4\}, & \overline{S} = \{2\}. \end{cases} \qquad V_{2j}^S(A) \begin{cases} S = \{2\}, & \overline{S} = \{\underline{\mathbf{1}}, 3, 4\}, \\ S = \{1,2\}, & \overline{S} = \{\underline{\mathbf{3}}, 4\}, \\ S = \{2,3\}, & \overline{S} = \{\underline{\mathbf{1}}, 4\}, \\ S = \{2,4\}, & \overline{S} = \{\underline{\mathbf{1}}, \underline{\mathbf{3}}\}, \\ S = \{1,2,3\}, & \overline{S} = \{4\}, \\ S = \{1,2,4\}, & \overline{S} = \{\underline{\mathbf{3}}\}, \\ S = \{2,3,4\}, & \overline{S} = \{\underline{\mathbf{1}}\}. \end{cases}$$

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$$V_{3j}^S(A)\begin{cases} S=\{3\}, & \overline{S}=\{1,2,4\},\\ S=\{1,3\}, & \overline{S}=\{2,4\},\\ S=\{2,3\}, & \overline{S}=\{\underline{\mathbf{1}},4\},\\ S=\{3,4\}, & \overline{S}=\{1,2\},\\ S=\{1,2,3\}, & \overline{S}=\{4\},\\ S=\{1,3,4\}, & \overline{S}=\{2\},\\ S=\{2,3,4\}, & \overline{S}=\{\underline{\mathbf{1}}\}.\end{cases} \qquad V_{4j}^S(A)\begin{cases} S=\{4\}, & \overline{S}=\{1,2,\underline{\mathbf{3}}\},\\ S=\{1,4\}, & \overline{S}=\{2,\underline{\mathbf{3}}\},\\ S=\{2,4\}, & \overline{S}=\{\underline{\mathbf{1}},\underline{\mathbf{3}}\},\\ S=\{3,4\}, & \overline{S}=\{1,2\},\\ S=\{1,2,4\}, & \overline{S}=\{\underline{\mathbf{3}}\},\\ S=\{1,3,4\}, & \overline{S}=\{2\},\\ S=\{2,3,4\}, & \overline{S}=\{\underline{\mathbf{1}}\}.\end{cases}$$

Besides, similarly to Algorithm 2, we present Algorithm 3 for identifying CKV-type matrices, which requires less computational cost than Algorithm 2.

Algorithm 3 A direct method for identifying CKV-type matrices.

Input. A matrix $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ with $N_-\neq\emptyset$.
Step 0. Compute N_-, and set $m:=|N_+|$, $N_- =: \{i_1,i_2,\dots,i_l\}$, where $l=|N_-|$.
Step 1. For $i\in N_-$, set $k=0$, $S_i^{(k)}=N_-$, $\overline{S_i^{(k)}}=N_+$, and go to Step 2.
Step 2. If $\overline{S_i^{(k)}}=\emptyset$, then '$A$ is not a CKV-type matrix', stop. Otherwise, compute

$$d_i^{(k)} := |a_{ii}| - r_i^{S_i^{(k)}}(A).$$

If $d_i^{(k)}\le 0$, then 'A is not a CKV-type matrix', stop. Otherwise, compute

$$\blacktriangle_i^{(k)} := \left\{ j\in\overline{S_i^{(k)}} : a_{jk}\neq 0 \text{ for some } k\in S_i^{(k)}\right\}$$

and

$$\widehat{\Theta}_i^{(k)} := \left\{ j\in\blacktriangle_i^{(k)} : \left(|a_{ii}| - r_i^{S_i^{(k)}}(A)\right)\left(|a_{jj}| - r_j^{\overline{S_i^{(k)}}}(A)\right) \le r_i^{\overline{S_i^{(k)}}}(A)r_j^{S_i^{(k)}}(A)\right\}.$$

If $\widehat{\Theta}_i^{(k)}=\emptyset$, then $S_i^{(k)}\in S_i^\star(A)$, i.e., $S_i^\star(A)\neq\emptyset$. Otherwise, go to Step 3.
Step 3. Set

$$S_i^{(k+1)} := S_i^{(k)}\cup\widehat{\Theta}_i^{(k)} \text{ and } \overline{S_i^{(k+1)}} := \overline{S_i^{(k)}}\setminus\widehat{\Theta}_i^{(k)},$$

and go to Step 2 (Replace k by $k+1$).
Output. A is either not a CKV-type matrix or a CKV-type matrix for $\{S_{i_1}^{(k_{i_1})}, S_{i_2}^{(k_{i_2})},\dots,S_{i_l}^{(k_{i_l})}\}$, where $k_{i_t}\in\{0,1,\dots,m\}$ with $t=1,2,\dots,l$.

EXAMPLE 3.11. Consider the matrix A_6 in Example 2.7 with the corresponding nonzero sparsity pattern being illustrated in Figure 3.4. Using Algorithm 2 and Algorithm 3, we get the numerical results given in Table 3.1. We also list the computation time of $\Theta_i^{(k)}$ in Algorithm 2 and of $\widehat{\Theta}_i^{(k)}$ in Algorithm 3 in Table 3.2, which determines the overall computational cost.

As can be seen from Table 3.1 and Table 3.2, for a given large-scale sparse matrix, Algorithm 3 requires less CPU time.

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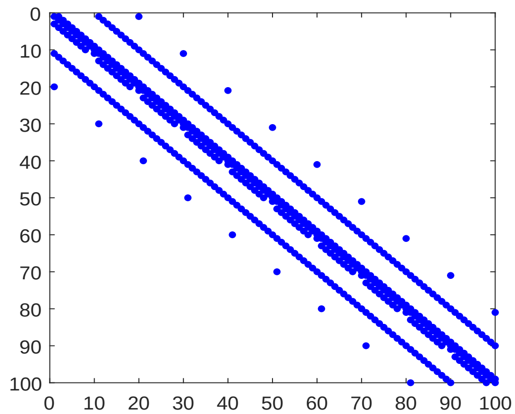

FIG. 3.4. *Nonzero sparsity pattern of the matrix* $A_6 \in \mathbb{R}^{100\times 100}$.

TABLE 3.1
Identifying whether a matrix is a CKV-type matrix or not by using Algorithms 2 and 3.

Order	Matrix	Algorithm	Result	CPU(s)
900	A_6	Algorithm 2 Algorithm 3	A_6 is a CKV type matrix	0.056807 0.051236
2500	A_6	Algorithm 2 Algorithm 3	A_6 is a CKV type matrix	0.377416 0.337005

TABLE 3.2
CPU time of $\Theta_i^{(k)}$ *and* $\widehat{\Theta}_i^{(k)}$ *for some* $i \in N_-$.

i	51	151	251	1001	2001	2401
Algorithm 2	0.000401	0.000326	0.000394	0.000513	0.000321	0.000297
Algorithm 3	0.000138	0.000139	0.000136	0.000169	0.000125	0.000139

4. Conclusions. In this paper, we first present a necessary condition for a CKV-type matrix as well as a sufficient condition for identifying when a matrix is not a CKV-type matrix and also give an equivalent condition for characterizing CKV-type matrices. Based on these criteria, we propose a direct algorithm to identify CKV-type matrices. Numerical examples show that the proposed results are efficient. In addition, we address three possible sparsity patterns in CKV-type matrices including S-SOB, S-SOB type-I, and S-SOB type-II matrices, and we prove that they are subclasses of CKV-type matrices. Moreover, we give a necessary and sufficient condition involving sparsity patterns for CKV-type matrices and obtain a direct algorithm requiring less computational cost for identifying CKV-type matrices. Besides, we analyze the relationships among SDD, DSDD, DZ, DZ-type, CKV, CKV-type, SOB, S-SOB, S-SOB type-I, and S-SOB type-II matrices.

Acknowledgements. The authors are grateful to the editor and the anonymous referees for their helpful suggestions on the quality improvement of our manuscript. We also

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thank Xinnian Song for his help with the program code during the revision stage of the manuscript. This work was partly supported by the National Natural Science Foundations of China (12061087 and 31600299), the Youth Science and the Technology Nova Program of Shaanxi Province (2022KJXX-01), the Scientific Research Program Funded by Yunnan Provincial Education Department (2022J0949), the Natural Science Foundations of Shaanxi province, China (2017JQ3020 and 2020JM-622).

Conflicts of interest. The authors declare that they have no conflict of interest.

Appendix A. The MATLAB code for Algorithm 2.

```
A; n=length(A);a=[1:n];r=zeros(n,1);
for i=1:n; r(i)=sum(abs(A(i,:)))-abs(A(i,i)); end
N_A=[];
for i=1:n
    if (abs(A(i,i))-r(i)<=0)
        N_A=[N_A,i];
    else
        continue
    end
end
S=cell(1,length(N_A));
for i=1:length(S); S{i}=N_A; end
for i=1:length(N_A)
    while (length(S{i})<n)
      m=find(S{i}==N_A(i));
      [S_bar,r_i_S_i,r_i_S_bar_i,r_j_S_i,r_j_S_bar_i]=
      parameter(r,n,a,A,S{i});
      d=abs(A(N_A(i),N_A(i)))-r_i_S_i(m);
      if (d<=0)
        fprintf('A is not a CKV-type matrix');
        return;
      end
      theta=[];
      for j=1:length(S_bar)
          e=d*(abs(A(S_bar(j),S_bar(j)))-r_j_S_bar_i(j))-
            r_i_S_bar_i(m)*r_j_S_i(j);
          if (e<=0)
              theta=[theta,S_bar(j)];
          else
              continue
          end
      end
      if (numel(theta)==0);
             S{i}=S{i};
         fprintf('Output S');
         break;
      else
         S{i}=union(S{i},theta);
      end
      if (numel(S{i})==n)
         fprintf('A is not a CKV-type matrix');
         return;
      else
         continue
      end
    end
end
```

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```
function [S_bar,r_i_S_i,r_i_S_bar_i,r_j_S_i,r_j_S_bar_i]
         =parameter(r,n,a,A,S)
n_S=length(S);
S_bar=setdiff(a,S);
r_i_S_i=zeros(1,n_S);
r_i_S_bar_i=zeros(1,n_S);
   for k=1:n_S
      r_i_S_bar_i(k)=sum(abs(A(S(k),S_bar)));
      r_i_S_i(k)=r(S(k))-r_i_S_bar_i(k);
   end
r_j_S_i=zeros(1,n-n_S);
r_j_S_bar_i=zeros(1,n-n_S);
   for k=1:n-n_S
      r_j_S_i(k)=sum(abs(A(S_bar(k),S)));
      r_j_S_bar_i(k)=r(S_bar(k))-r_j_S_i(k);
   end
end
```

Appendix B. Proof of Propostiion 3.7.

Proof. Note that $N_- := \{i \in N : |a_{ii}| \le r_i(A)\}$. If $N_- = \emptyset$, then A is an SDD matrix. For this case, the conclusion is obviously true. If $N_- \neq \emptyset$, then for any $i_0 \in N_-$, we have $i_0 \in S \cup \overline{S}$.

The first case. If $i_0 \in \overline{S}$, then since A is an S-SOB type-II matrix, it follows that for each $j \in S$,

$$\begin{aligned} |a_{i_0,i_0}| &> r_{i_0}^{\overline{S}}(A) && \text{and} \\ (|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))|a_{jj}| &> r_{i_0}^{S}(A) r_j(A) && \text{if } a_{i_0,j} \neq 0. \end{aligned}$$

Without loss of generality, we assume that $S = \{j_1, \ldots, j_l\}$ and $\overline{S} = \{i_0, i_1, \ldots, i_k\}$. Let $S' := \{i_0, i_1, \ldots, i_k, j_1, \ldots, j_k\}$ and $\overline{S'} := \{j_{k+1}, \ldots, j_l\}$, where $a_{i_0,j} = 0$ for all $j \in \{j_1, \ldots, j_k\}$ and $a_{i_0,j} \neq 0$ for all $j \in \overline{S'}$. Then, we easily obtain that

$$\begin{aligned} |a_{i_0,i_0}| &> r_{i_0}^{S'}(A) && \text{and} \\ (|a_{i_0,i_0}| - r_{i_0}^{S'}(A))|a_{jj}| &> r_{i_0}^{\overline{S'}}(A) r_j(A). \end{aligned} \tag{B.1}$$

According to $|a_{i_0,i_0}| \le r_{i_0}(A)$, it follows from (3.3) and (B.1) that

$$(|a_{i_0,i_0}| - r_{i_0}^{S'}(A))(|a_{jj}| - r_j^{\overline{S'}}(A)) > r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A).$$

This means that $S' \in S_{i_0}^{\star}(A)$, that is, $S_{i_0}^{\star}(A) \neq \emptyset$.

The second case. Suppose that $i_0 \in S$. If $S = \{i_0\}$, then for each $j \in N \setminus \{i_0\}$,

$$r_j^{\overline{S}}(A) = r_j(A) - |a_{j,i_0}| = r_j^{i_0}(A) \qquad \text{and} \qquad r_j^{S}(A) = |a_{j,i_0}|.$$

It follows that

$$\begin{aligned} |a_{jj}| &> r_j^{i_0}(A) && \text{and} \\ (|a_{jj}| - r_j^{i_0}(A))|a_{i_0,i_0}| &> |a_{j,i_0}| r_{i_0}(A) && \text{if } a_{j,i_0} \neq 0, \end{aligned}$$

which implies that A is a DZ matrix, and thus a CKV-type matrix.

If $S \neq \{i_0\}$, then without loss of generality we assume that

$$S = \{i_0, i_1, \ldots, i_k\} \quad \text{and} \quad \overline{S} = \{j_1, j_2, \ldots, j_l\}.$$

By Definition 3.6, it follows that for $i_0 \in S$ and all $j \in S \setminus \{i_0\}$,

$$|a_{i_0,i_0}| > r_{i_0}^{\overline{S}}(A) \quad \text{and} \tag{B.2}$$
$$(|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))|a_{jj}| > r_{i_0}^{S}(A) r_j(A) \quad \text{if } a_{i_0,j} \neq 0.$$

By (B.2), it follows that there is $i_t \in S \setminus \{i_0\}$ such that $a_{i_0,i_t} \neq 0$. Otherwise, if $a_{i_0,j} = 0$ for all $j \in S \setminus \{i_0\}$, then

$$|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A) = |a_{i_0,i_0}| - r_{i_0}(A) \leq 0,$$

which contradicts (B.2). Let

$$S' := \{i_0, i_1, \cdots, i_l, j_1, \ldots, j_l\} \qquad \text{and} \qquad \overline{S'} := \{i_{l+1}, \ldots \ldots, i_k\},$$

where $a_{i_0,j} = 0$ for all $j \in \{i_1, \ldots, i_l\}$ and $a_{i_0,j} \neq 0$ for all $j \in \overline{S'}$. Then, we easily obtain that

$$|a_{i_0,i_0}| > r_{i_0}^{S'}(A), \quad \text{and} \tag{B.3}$$
$$(|a_{i_0,i_0}| - r_{i_0}^{S'}(A))|a_{jj}| > r_{i_0}^{\overline{S'}}(A) r_j(A) \quad \text{for all } j \in \overline{S'}. \tag{B.4}$$

According to $|a_{i_0,i_0}| \leq r_{i_0}(A)$, it follows from (3.3), (B.3), and (B.4) that

$$(|a_{i_0,i_0}| - r_{i_0}^{S'}(A))(|a_{jj}| - r_j^{\overline{S'}}(A)) > r_{i_0}^{\overline{S'}}(A) r_j^{S'}(A).$$

This means that $S' \in S_{i_0}^{\star}(A)$, that is, $S_{i_0}^{\star}(A) \neq \emptyset$. The conclusion follows from the above two cases. $\square$

Appendix C. Relationships between some subclasses of H-matrices.

In this section, we discuss the relationships among SDD, DSDD, DZ, DZ-type, CKV (Σ-SDD), CKV-type, SOB, S-SOB, S-SOB type-I, and S-SOB type-II matrices. Before that we recall the definitions of DZ-type and SOB matrices.

DEFINITION C.1 ([32]). *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a Dashnic-Zusmanovich-type (DZ-type) matrix if for each $i \in N$, there exists $j \in N, j \neq i$, such that (3.5) holds.*

DEFINITION C.2 ([18]). *Let S be a nonempty proper subset of N and $\overline{S} = N \setminus S$. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an SOB-matrix if for all $i \in S, j \in \overline{S}$,*

$$(|a_{ii}| - r_i^{S}(A))|a_{jj}| > r_i^{\overline{S}}(A) r_j(A), \tag{C.1}$$

and

$$(|a_{jj}| - r_j^{\overline{S}}(A))|a_{ii}| > r_j^{S}(A) r_i(A). \tag{C.2}$$

As shown in [18], the SOB class belongs to the S-SOB class and contains the DSDD class, that is,

$$\{\text{DSDD}\} \subseteq \{\text{SOB}\} \subseteq \{S\text{-SOB}\}.$$

C.1. The relationships among SOB, S-SOB, S-SOB type-I, and S-SOB type-II matrices.

PROPOSITION C.3. *If A is an SOB matrix, then A is an S-SOB type-II matrix, that is,*

$$\{\textit{SOB}\} \subseteq \{S\textit{-SOB type-II}\}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Proof. Since A is an SOB matrix, it follows that for all $i \in S, j \in \overline{S}$, (C.1) and (C.2) hold. We next divide our proof into two cases.

Case I. Suppose $|a_{ii}| > r_i(A)$ for all $i \in S$. For this case, we see that for all $i, j \in S$ and $i \neq j$,

$$(|a_{ii}| - r_i^{\overline{S}}(A))|a_{jj}| > r_i^S(A) r_j(A),$$

which together with (C.1) and (C.2) implies that

$$\begin{aligned} |a_{ii}| &> r_i^{\overline{S}}(A) && \text{for all } i \in N \quad \text{and} \\ (|a_{ii}| - r_i^{\overline{S}}(A))|a_{jj}| &> r_i^S(A) r_j(A) && \text{for } j \in S,\ i \in N \setminus \{j\}. \end{aligned}$$

Therefore, from Definition 3.6 it holds that A is an S-SOB type-II matrix.

Case II. Suppose $|a_{i_0,i_0}| \leq r_{i_0}(A)$ for some $i_0 \in S$. According to (C.1), it follows that for all $j \in \overline{S}$,

$$|a_{jj}| > r_j(A).$$

Similarly to the proof of Case I, we can prove that A is an S-SOB type-II matrix. The conclusion follows from Case I and Case II. □

EXAMPLE C.4. Consider the following matrices:

$$A_7 = \begin{bmatrix} 3 & -1 & -3 & 0 \\ -1 & 3 & 0 & -8 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \qquad A_8 = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.2 \\ 0.3 & 1 & 0.5 & 0.5 \\ 0.65 & 0 & 1 & 0 \\ 0 & 0.3 & 0 & 1 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 1 & 0.25 & 0.25 & 0 & 0.5 \\ 0.52 & 1 & 0.1 & 0 & 0.6 \\ 0.2 & 0.3 & 1 & 0.2 & 0 \\ 0 & 0 & 0.7 & 1 & 0.5 \\ 0 & 0 & 0 & 0.6 & 1 \end{bmatrix}.$$

It is easy to validate that A_7 is an S-SOB type-I matrix for $S = \{1, 3\}$ and also an S-SOB type-II matrix for $S = \{3, 4\}$ but not an S-SOB matrix, consequently, not a DSDD matrix. On the other hand, by calculations, we know that A_8 is an SOB matrix and an S-SOB type-II matrix for $S = \{3, 4\}$ but not an S-SOB type-I matrix and that A_9 is an S-SOB matrix for $S = \{4, 5\}$ and an S-SOB type-I matrix for $S = \{2, 5\}$ but not an S-SOB type-II matrix. This means that S-SOB and S-SOB type-II matrices are not necessarily S-SOB type-I matrices and that S-SOB and S-SOB type-I matrices are not necessarily S-SOB type-II matrices. Therefore, the relations among SOB, S-SOB, S-SOB type-I, and S-SOB type-II matrices can be depicted as follows:

- $\{S$-SOB type-I$\} \nsubseteq \{S$-SOB$\}$, $\{S$-SOB$\} \nsubseteq \{S$-SOB type-I$\}$
- $\{S$-SOB type-II$\} \nsubseteq \{S$-SOB$\}$, $\{S$-SOB$\} \nsubseteq \{S$-SOB type-II$\}$
- $\{S$-SOB type-I$\} \nsubseteq \{$SOB$\}$, $\{$SOB$\} \nsubseteq \{S$-SOB type-I$\}$
- $\{S$-SOB type-I$\} \nsubseteq \{S$-SOB type-II$\}$, $\{S$-SOB type-II$\} \nsubseteq \{S$-SOB type-I$\}$.

EXAMPLE C.5. Consider the matrix

$$A_{10} = \begin{bmatrix} 1 & 0.5 & 0.25 & 0.25 \\ 0.5 & 1 & 0.25 & 0.25 \\ 0.25 & 0.25 & 1 & 0.5 \\ 0 & 0 & 0.8 & 1 \end{bmatrix}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

It is easy to verify that A_{10} is a DZ matrix but not an S-SOB type-I matrix. By the matrix A_7 in Example C.4, it follows that A_7 is an S-SOB type-I matrix but not a DZ matrix. This implies that the set of all DZ matrices is not a subset of the set of all S-SOB type-I matrices, and the set of all S-SOB type-I matrices is also not a subset of the set of all DZ matrices, that is,

$$\{\text{DZ}\} \not\subseteq \{S\text{-SOB type-I}\} \quad \text{and} \quad \{S\text{-SOB type-I}\} \not\subseteq \{\text{DZ}\}.$$

However, by Remark 3.5, it follows that a DSDD matrix is an S-SOB type-I matrix. This means that the set of all DSDD matrices is a subset of the intersection of the set of all S-SOB type-I matrices and the set of all DZ matrices, that is,

$$\{\text{DSDD}\} \subseteq \{S\text{-SOB type-I}\} \cap \{\text{DZ}\}.$$

C.2. The relationships among SOB, S-SOB, and CKV matrices.

PROPOSITION C.6. *If A is an SOB matrix, then A is a CKV matrix, that is,*

$$\{SOB\} \subseteq \{CKV\}.$$

Proof. We divide our proof into two cases.

Case I. Suppose $|a_{ii}| > r_i(A)$ for all $i \in N$. For this case, A is an SDD matrix, and the conclusion follows obviously.

Case 2. Suppose $|a_{i_0,i_0}| \le r_{i_0}(A)$ for some $i_0 \in N = S \cup \overline{S}$. If $i_0 \in S$, then it follows from A being an SOB matrix that for all $j \in \overline{S}$,

$$\begin{aligned}(|a_{i_0,i_0}| - r_{i_0}^{S}(A))|a_{jj}| &> r_{i_0}^{\overline{S}}(A)r_j(A) = r_{i_0}^{\overline{S}}(A)r_j^{\overline{S}}(A) + r_{i_0}^{\overline{S}}(A)r_j^{S}(A)\\ &\ge (|a_{i_0,i_0}| - r_{i_0}^{S}(A))r_j^{\overline{S}}(A) + r_{i_0}^{\overline{S}}(A)r_j^{S}(A).\end{aligned}$$

Therefore,

$$(|a_{i_0,i_0}| - r_{i_0}^{S}(A))(|a_{jj}| - r_j^{\overline{S}}(A)) > r_{i_0}^{\overline{S}}(A)r_j^{S}(A).$$

If $i_0 \in \overline{S}$, then it follows from A being an SOB matrix that for all $i \in S$,

$$\begin{aligned}(|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))|a_{ii}| &> r_{i_0}^{S}(A)r_i(A) = r_{i_0}^{S}(A)r_i^{S}(A) + r_{i_0}^{S}(A)r_i^{\overline{S}}(A)\\ &\ge (|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A))r_i^{S}(A) + r_i^{\overline{S}}(A)r_{i_0}^{S}(A).\end{aligned}$$

It follows that

$$(|a_{ii}| - r_i^{S}(A))(|a_{i_0,i_0}| - r_{i_0}^{\overline{S}}(A)) > r_i^{\overline{S}}(A)r_{i_0}^{S}(A).$$

Hence, the conclusion follows from Case I and Case II. ∎

Next we give an example to show that neither of the two classes, S-SOB and CKV matrices, is a subset of the other one.

EXAMPLE C.7. Consider again the matrix A_9 in Example C.4. It is easy to verify that A_9 is an S-SOB matrix for $S = \{4, 5\}$ but not a CKV matrix. On the other hand, consider the matrix

$$A_{11} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0.5 \\ 1 & 2 & 0.33 & 1 \end{bmatrix}.$$

By calculation, we know that A_{11} is a CKV matrix for $S = \{1, 2\}$ but not an S-SOB matrix. Hence, the relationship between S-SOB and CKV matrices is

$$\{\text{CKV}\} \not\subseteq \{S\text{-SOB}\} \quad \text{and} \quad \{S\text{-SOB}\} \not\subseteq \{\text{CKV}\}.$$

C.3. The relationships among CKV, S-SOB type-I, and S-SOB type-II matrices.
EXAMPLE C.8. Consider again the matrices A_7 and A_8 in Example C.4 and A_{11} in Example C.7. We know that A_7 is an S-SOB type-I matrix and an S-SOB type-II matrix but not a CKV matrix and that A_8 is a CKV matrix for $S = \{1, 2\}$ but not an S-SOB type-I matrix and that A_{11} is a CKV matrix for $S = \{1, 2\}$ but not an S-SOB type-II matrix. Hence, the relations among CKV, S-SOB type-I, and S-SOB type-II matrices can be given by

- $\{S\text{-SOB type-I}\} \not\subseteq \{\text{CKV}\}$, $\{\text{CKV}\} \not\subseteq \{S\text{-SOB type-I}\}$
- $\{S\text{-SOB type-II}\} \not\subseteq \{\text{CKV}\}$, $\{\text{CKV}\} \not\subseteq \{S\text{-SOB type-II}\}$.

C.4. The relationship between S-SOB type-I (type-II) matrices and DZ-type matrices.
EXAMPLE C.9. As shown in [32], a DSDD matrix is not necessarily a DZ-type matrix, and the class of DSDD is a subclass of S-SOB type-I matrices by Remark 3.5. This means that an S-SOB type-I matrix is not necessarily a DZ-type matrix. On the other hand, consider the matrix A_{10} in Example C.5. We have that A_{10} is a DZ-type matrix but not an S-SOB type-I matrix. This shows that neither of the two classes, S-SOB type-I and DZ-type matrices, is a subset of the other one, that is,

$$\{S\text{-SOB type-I}\} \not\subseteq \{\text{DZ-type}\} \quad \text{and} \quad \{\text{DZ-type}\} \not\subseteq \{S\text{-SOB type-I}\},$$

which imply that

$$\{\text{CKV-type}\} \not\subseteq \{S\text{-SOB type-I}\}.$$

Consider the matrices A_8 and A_9 in Example C.4. We know that A_8 is an S-SOB type-II matrix for $S = \{3, 4\}$ but not a DZ-type matrix, and A_9 is a DZ-type matrix but not an S-SOB type-II matrix. This implies that neither of the two classes, S-SOB type-II and DZ-type matrices, is a subset of the other one, that is,

$$\{S\text{-SOB type-II}\} \not\subseteq \{\text{DZ-type}\} \qquad \text{and} \qquad \{\text{DZ-type}\} \not\subseteq \{S\text{-SOB type-II}\},$$

which imply that

$$\{\text{CKV-type}\} \not\subseteq \{S\text{-SOB type-II}\}.$$

C.5. The relationship between S-SOB matrices and DZ (DZ-type) matrices.
EXAMPLE C.10. Consider again the matrices A_7 and A_8 in Example C.4. It is easy to verify that A_7 is a DZ-type matrix but not an S-SOB matrix and that A_8 is an S-SOB matrix but not a DZ-type matrix and a DZ matrix. On the other hand, consider the following matrix:

$$A_{12} = \begin{bmatrix} 1 & 0.1 & 0 & 0.5 \\ 0.1 & 1 & 0 & 0.5 \\ 0.1 & 0 & 1 & 0.5 \\ 0.4 & 1 & 0.3 & 1 \end{bmatrix}.$$

By calculation, it follows that A_{12} is a DZ matrix but not an S-SOB matrix. This shows that neither of the two classes, S-SOB and DZ-type (DZ) matrices, is a subset of the other one, that is,

$$\{\text{DZ-type}\} \not\subseteq \{S\text{-SOB}\} \quad \text{and} \quad \{S\text{-SOB}\} \not\subseteq \{\text{DZ-type}\},$$

and

$$\{\text{DZ}\} \not\subseteq \{S\text{-SOB}\} \quad \text{and} \quad \{S\text{-SOB}\} \not\subseteq \{\text{DZ}\}.$$

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